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TEST PAPERS AND SOLUTIONS
ON ELECTRICAL ENGINEERING

THE COMPANION VOLUME TO THIS BOOK

By the Same Author

"PRINCIPLES OF ELECTRICAL ENGINEERING"

The problems which are solved in "Test Papers and Solutions on Electrical Engineering" are all chosen and arranged with a view to emphasising the different aspects of principles dealt with in the various chapters of the above work. "Principles of Electrical Engineering" is, therefore, the necessary preliminary of the two books, which are issued separately for convenience of reference The chapter headings are the same The book aims to give a comprehensive account of the basic principles of the science of electrical engineering, emphasis being placed on the identity of the principles relating to both heavy-current and light-current engineering practice

Attention is given to simplification in methods of solution of a wide range of technical problems, and new developments in applications, testing, and materials, are taken into account.

TEST PAPERS AND SOLUTIONS

ON

ELECTRICAL ENGINEERING

The Companion Volume to Dr. Wall's "Principles of Electrical Engineering," Covering the Requirements of the B.Sc. (Engineering), A.M.I.E.E. and Higher Examinations in the Principles of the Subject

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WITH DETAILED SOLUTIONS OF 203 PROBLEMS
AND 371 DIAGRAMS

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PREFACE

" The beginning of science is measurement " is a well-established truth, and the working out of numerical problems in electrical engineering can become both a fascinating and a profitable occupation. A competent electrical engineer can predict with confidence and accuracy the detailed performance of an electrical appliance before any part of it has been constructed, and this is possible without having to resort to the use of "factors of safety" : it is questionable whether this can be done to anything like the same degree in any other branch of applied science. There is no more certain road to such competence than by continual practice in the solving of all kinds of examples in applied electricity and magnetism.

Several of the examples given in this book extend somewhat beyond the range of the text of the parent volume " Principles of Electrical Engineering," and in such cases the reader can with advantage search the technical journals for the missing details before turning to the solutions given in this book. By such means, a familiarity with the technical literature, its scope, and its utility, will be achieved, which will probably surprise the reader by the insight which it gives of its possibilities, and may lure him on to further study of what is, at present, almost unknown country to far too many electrical engineers.

To an extent which has been found reasonably possible, the whole range of applied electricity and magnetism has been drawn upon for the selection of the examples given, and both light-current and heavy-current problems have been considered with equal freedom in order to show the universality of the principles. Great care has been taken to check the numerical data of each step in the solutions for the individual examples, but it is probably too much to hope for that no slip or misprint has remained undetected, and information from readers who have been able to discover any such flaws will be welcomed.

T. F. W.

SHEFFIELD, 1946.

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TEST PAPERS AND SOLUTIONS ON ELECTRICAL ENGINEERING

Chapter I

FUNDAMENTAL UNITS : TECHNICAL UNITS

1.—*Express in joules the kinetic energy (i) of translation, (ii) of rotation.*

ANSWER.—The kinetic energy of translation

$= \frac{1}{2}mv^2$ ergs : for m in gm., v in cm./sec.

$= \frac{1}{2}mv^2 10^{-7}$ joules : for m in gm., v in cm./sec.

$= \frac{1}{2}M \times 10^3 (V \times 10^2)^2 10^{-7}$ joules · for M in kg., V in m./sec.

$= \frac{1}{2}MV^2$ joules : for M in kg., V in m./sec.

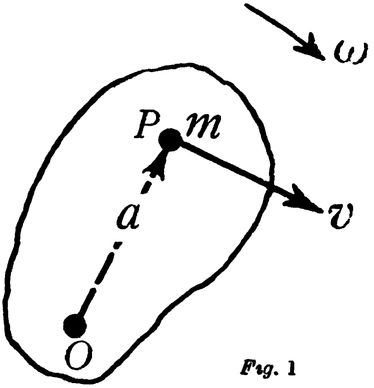


Fig. 1

The kinetic energy of rotation about the axis O for the elementary mass of m gm. at P in Fig. 1, is

$= \frac{1}{2}mv^2$ ergs : for v in cm./sec.

$= \frac{1}{2}m(a\omega)^2 10^{-7}$ joules · for ω in radians/sec. and a in cm.

$= \frac{1}{2}\omega^2 10^{-7} ma^2$ joules,

so that the kinetic energy of rotation about the axis O of the whole mass is

$= \frac{1}{2}\omega^2 10^{-7} \Sigma ma^2$ joules : for m in gm. and a in cm.

$= \frac{1}{2}J\omega^2$ joules : J in kg.-m.² units is the moment of inertia of the mass about O.

2.—*A train of mass G metric tons is travelling at a speed of v km./hour. Find the kinetic energy, assuming that the rotational energy is 25 per cent. of that of translation.*

ANSWER.—Kinetic energy of translation $= \frac{1}{2}MV^2$ joules, for M kg. and V m/sec. G metric tons $= 1{,}000\ G$ kg. : v km./hour $= 0{\cdot}279\ v$ m/sec

Kinetic energy of translation

$= \frac{1}{2}v^2 78G$ joules (watt-sec.)

$= 10{\cdot}8Gv^2 10^{-6}$kWh. [G in tons. v in km./hour.]

Total kinetic energy $= 1{\cdot}25$ (K.E. of translation) $= 13{\cdot}5Gv^2 10^{\ 6}$kWh

Thus, if $G = 50$ metric tons : $v = 100$ km./hour : Total K.E. $= 6{\cdot}7$ kWh.

3.—What driving force is necessary to produce an acceleration of A km./hour/sec. of a train on the level, neglecting the frictional resistance?

Answer.—Force in dynes = (mass in gm.) × (acceleration in cm./sec.^{2}) 28·3 × (G in metric tons) × (A in km./hr./sec.) for translation acceleration. The total accelerating force inclusive of the rotary motion = 1·25 × 28·3 × G × A = 35 GA kg. Thus, if A = 0·6 to 1 km./hr./sec., the accelerating force is from 21 to 35 kg./ton.

4.—An electrically operated vehicle of mass M kg. is running on the level at a speed of V metres/sec. If the vehicle is braked by the application of a constant torque, show by means of a diagram the relationship between speed and time, and between kinetic energy and time, during the braking period.

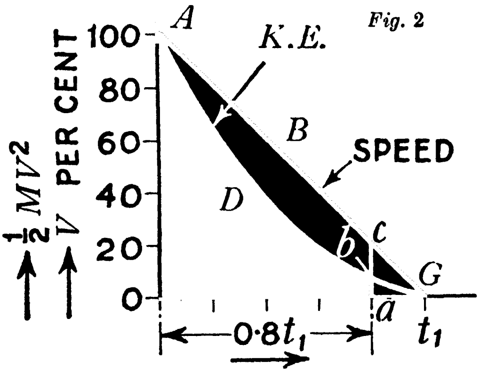


Fig. 2

Answer.—When a constant braking torque is applied, the retardation will be constant so that the (speed/time) relationship will be given by the straight line ABG in Fig. 2. At any moment during the braking period, at which the velocity is V m./sec. the kinetic energy will be $\frac{1}{2}MV^2$ joules, where M kg. is the mass, so that the (kinetic energy/time) relationship will be given by the parabola curve ADG. If the total time required to bring the vehicle to rest is t_0 sec. then in the time 0·8t_0 sec. the speed will have fallen to 0·2 of its initial value, so that the kinetic energy still stored in the moving vehicle will then be

$$\tfrac{1}{2}M(0{\cdot}2V)^2 = \tfrac{1}{2}(0{\cdot}04MV^2) \text{ joules.}$$

That is to say, when the speed has fallen to 20 per cent. of its initial value, the kinetic energy still remaining will be only 4 per cent. of its initial value. If, then, electric braking is used to reduce the speed to about 20 per cent. of its initial value and the vehicle is brought to rest by mechanical brakes, the demand on the brakes will be very small and, consequently, the cost of maintenance will be correspondingly reduced. If mechanical brakes are exclusively used, however, the wear on the friction surfaces is very great and the cost of maintenance high, whilst the powdered dust developed by the friction wear may percolate into the rotor structure and seriously impair the reliability of the insulation.

5.—A train running at a speed of 60 miles/hour on the level is brought to rest in 1,000 yards. Find what uniform resistance must be applied to the train.

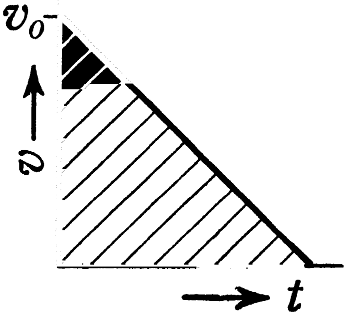


Fig. 3

ANSWER.—Speed of 60 miles/hr. = 2,680 cm./sec. The retardation will therefore be $a = \frac{2{,}680}{t}$ cm./sec.^{2} where t sec. is the time necessary to stop the train. The distance travelled is given by the shaded area of Fig. 3, that is, $S = \frac{1}{2}v_0t = 3{,}000 \times 12 \times 2{\cdot}54$ cm., so that $t = 68$ sec. The retarding force is therefore $F = M\frac{2{,}680}{68 \times 981}$ gm. for M in gm. or

$$F = 40 \text{ kg./ton} = 88 \text{ lb./ton}.$$

6.—At what speed will an engine of 500 horse-power take a train of 300 tons weight up an incline of 1 : 80, if the frictional resistance is 15 lb. weight per ton.

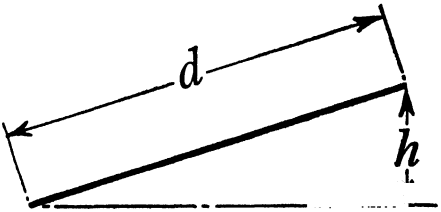


Fig. 4

ANSWER.—Assume in Fig. 4 that h is, say, 100 ft., so that d, the distance travelled, will be 8,000 ft. Then the work done against gravity will be $W_G = 300 \times 100 = 30{,}000$ ft.-tons ; work done against the frictional resistances will be $\frac{300 \times 15 \times 8{,}000}{2{,}240} =$ 16,050 ft.-tons. so that the total work done will be 46,050 ft.-tons.

If t sec. are required for the journey of 8,000 ft., then

$$\frac{46{,}050 \times 2{,}240}{t \times 550} = 500 \text{ H.P.}$$

so that $t = 376$ sec. and, consequently, the speed of travel up the incline will be

$$v = \frac{8{,}000 \times 3{,}600}{376 \times 5{,}280} = 14{\cdot}5 \text{ miles per hour.}$$

7.—A railway waggon weighing 30 tons strikes the buffers at a speed of 5 miles/hour on the level, and comes to rest in 1 foot. Find the mean resistance exerted by the buffer springs.

ANSWER.—5 miles per hour = 2·2 metres per second : Mass of waggon is $\frac{30 \times 2{,}240}{2{,}200} \times 1{,}000 = 30{,}500$ kg. Kinetic energy of the waggon $= \frac{1}{2} MV^2$ joules $= \frac{1}{2} \times 30{,}500 \times 2{\cdot}2^2$ joules = 54,500 ft.-lbs. The mean force exerted by the buffer springs is then $\frac{54{,}500}{1 \times 2{,}240} = 24{\cdot}8$ tons weight.

8.—The specific heat of copper is 0.094 $\frac{\text{kcal.}}{\text{kg. °C.}}$ and the density is 8·9. Express the specific heat in terms of $\frac{\text{joules}}{\text{c.cm. °C.}}$

ANSWER.—1 joule = 0·239 calorie : 1 calorie = 4·18 joules 1 kcal. = 4,180 joules : 1 kg. copper has a volume $\frac{1{,}000}{8{\cdot}9} = 112$ c.cm., so that the specific heat of copper is

$$0{\cdot}094 \frac{\text{kcal.}}{\text{kg.} : \text{°C.}} = \frac{0{\cdot}094 \times 4{,}180}{112} \left(\frac{\text{joules}}{\text{c.cm.} : \text{°C.}}\right) = 3{\cdot}5 \left(\frac{\text{joules}}{\text{c.cm.} : \text{°C.}}\right)$$

9.—Express the c.g.s. unit of mass in terms of joules $\frac{\text{sec.}^2}{\text{cm.}^2}$.

ANSWER.—1 dyne = 1 erg/cm. = 1 gm. $\frac{\text{cm.}}{\text{sec.}^2} = \frac{1 \text{ kg.}}{10^3} \frac{\text{cm.}}{\text{sec.}^2}$

$$1 \text{ dyne} = 10^{-7} \frac{\text{joule}}{\text{cm.}} = 1 \frac{\text{kg.}}{10^3} \frac{\text{cm.}}{\text{sec.}^2}$$

$$1 \text{ kg.} = \frac{\text{joule sec.}^2}{10^4 \text{ cm.}^2}$$

10.—Express the c.g.s. unit of force in terms of joules/metre.

ANSWER.—981 dynes = weight of 1 gm. = weight of $\frac{1 \text{ kg.}}{10^3}$

$$1 \text{ joule} = 10^7 \text{ ergs} = 10^7 \text{ dyne-cms.}$$

$$1 \text{ joule/m.} = 10^5 \text{ dynes} = \frac{1{\cdot}02}{10} \text{ kg.}$$

$$9{\cdot}81 \frac{\text{joule}}{\text{m.}} = 1 \text{ kg.}$$

11.—If the cost of coal is 12s. per ton and the heat value is 3,000 kcal. per lb., and if 4,500 kcal. are required for the generation of 1 kWh., find (i) the thermal efficiency of generation, and (ii) the cost of generation per kWh.

ANSWER.—(i) Efficiency $= \frac{860}{4{,}500} = 19{\cdot}2$ per cent.

(ii) The total cost of generation involves the fixed charges, the cost of fuel, and the number of hours per annum during which generation takes place,

that is, $K = \frac{a \cdot p}{h} + b$

where, h is the number of hours per annum during which generation takes place,

a is the capital cost per kW of installed plant,

p is the percentage for interest, depreciation, repairs, management,

b is the cost of fuel in pence per kWh.

If for example, $a = £20 = 4{,}800$ pence per kW. of installed plant,

$p = 14$ per cent., that is, $0{\cdot}14$,

$$b = \left(\frac{4{,}500}{3{,}000}\right)\left(\frac{144}{2{,}240}\right)$$
$$= 0{\cdot}096 \text{ penny per kWh.}$$

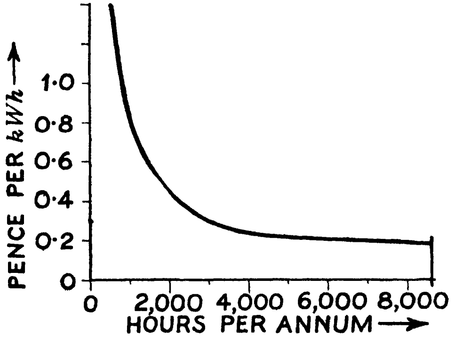


Fig 5.

so that $$K = \frac{0{\cdot}14 \times 4{,}800}{h} + 0{\cdot}096$$

that is, $$K = \left(\frac{670}{h} + 0{\cdot}096\right) \text{ penny per kWh. generated.}$$

In Fig. 5 this relationship between K and h is shown. It is to be noted that in the case of a water-power station the factor $b = 0$, but the capital costs are higher than for a steam-driven station.

12.—In a fuel-fired central heating plant 1 lb. of coal of calorific value 3,500 kcal. per lb. can produce about 0·9 kWh. of useful heat energy. Find the thermal efficiency of the plant. A fuel-fired boiler in continuous

service can produce about 2·25 kWh. of useful heat energy per lb. of coal. Find the thermal efficiency.

ANSWER.—Since 1 kWh. = 860 kcal., the efficiency of the central heating plant is

$$\frac{0{\cdot}9 \times 860}{3{,}500} = 22 \text{ per cent.}$$

For the fuel-fired boiler in continuous service the efficiency is

$$\frac{2{\cdot}25 \times 860}{3{,}500} \simeq 55 \text{ per cent.}$$

In comparison with these results it is to be noted that the efficiency ot an electric boiler of average size for continuous service is about 97 per cent.

Chapter II

STRUCTURE OF THE ATOM : CONDUCTORS AND INSULATORS : THE ELECTRIC CURRENT : ELECTRIC RESISTANCE

1.—Describe the constructional features of a dry (e.g. torch) battery.

Answer.—Such a cell is a form of the Leclanché Cell, and is shown in section in Fig. 1. The carbon rod forms the positive pole and the zinc container the negative pole. When a current is being taken from the cell the direction of current flow will be from the zinc container to the carbon rod, and as it passes through the ammonium chloride paste (NH_4Cl) it dissociates this substance and releases hydrogen. If no special means were provided to trap the hydrogen it would travel to the carbon rod and form an insulating layer on its surface. Further, since the hydrogen is electro-positive, it would produce an e.m.f. directly opposed to the e.m.f. of the cell, this phenomenon being known as the " polarisation " of the cell. The layer of manganese dioxide (MnO_2), shown in the diagram, enables the hydrogen to unite with the oxygen to form water. If the cell discharges at a heavy rate, then the hydrogen becomes generated more quickly than the manganese dioxide can absorb it so that some hydrogen will reach the carbon rod, thus producing some degree of polarisation and consequent reduction of the e.m.f. The normal e.m.f. of such a cell is about 1·5 volts.

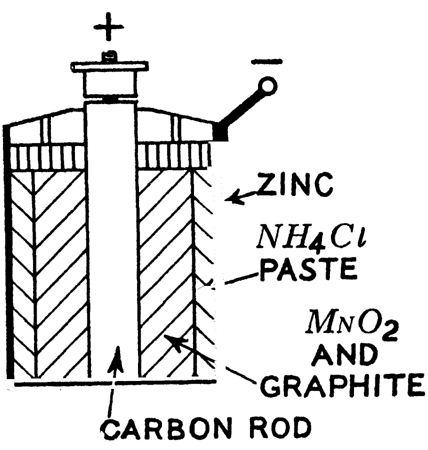


Fig. 1.

2.—With the aid of a diagram of connections, show how the internal resistance of a cell may be measured.

Answer.—A diagrammatical sketch of the circuit connections is shown in Fig. 2 in which *G* is a steady source of d.c. pressure and *PW* is a wire of a cross section which is uniform and such that the length *PW* is about 1 metre, the pressure drop V_1 for this length being about 25 per cent. greater than the e.m.f. *E* of the cell *K* which is under test.

The cell *K* is connected to *P* as shown, and, through a switch *S*, to a resistance R_2 of known value. One terminal of a galvanometer or other detector *D* is connected to the negative pole of *K* and the other terminal of the detector goes to a sliding contact *T* on the wire *PW*. If the switch *S* is open and the sliding contact adjusted so that *D* shows no deflection, then if the length of the portion *PT* of the wire is l_o cms. and the whole length *L* cms., it is easily seen that

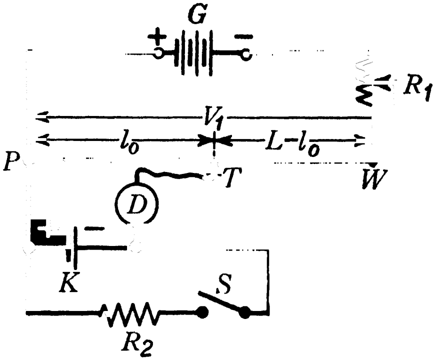


Fig. 2.

$$\frac{l_o}{L} = \frac{E}{V_1} \quad \ldots\ldots \quad \text{(i)}$$

If the switch *S* is now closed and the contact *T* readjusted so that the detector again shows no deflection, the length *PT* of the wire *PW* now being l_c cms. then,

$$\frac{l_c}{L} = \frac{V_k}{V_1} \quad \ldots\ldots \quad \text{(ii)}$$

where V_k is the p.d. across the terminals of the cell *K*. But,

$$V_k = IR_2 \quad \ldots\ldots \quad \text{(iii)}$$

where *I* amperes is the current flowing from the cell through the resistance R_2. If R_k ohms is the required value of the internal resistance of the cell *K*, then

$$E = V_k + IR_k \quad \ldots\ldots \quad \text{(iv)}$$

From equations (iii) and (iv)

$$E = V_k\left(1 + \frac{R_k}{R_2}\right) \quad \ldots\ldots \quad \text{(v)}$$

and from (i) and (ii)

$$\frac{l_o}{l_c} = \frac{E}{V_k} \quad \ldots\ldots \quad \text{(vi)}$$

Eliminating *E* between (v) and (vi) then gives

$$R_k = R_2\left(\frac{l_o - l_c}{l_c}\right)$$

The arrangement shown in Fig. 2 of a graduated resistance wire *PW* of uniform cross section in combination with a sliding contact is generally known as a " potentiometer slide wire."

3.—If the resistance of wire used for a telephone line is 55 ohms per mile and its conductivity is 56 siemens/metre/mm.², find the cross-sectional area.

ANSWER.—The resistance R of the wire is given by the expression,

$$R = P\frac{l}{q}$$

where $R = 55$ ohms for one mile of the wire, $l = 1{,}610$ metres, and $V = \frac{1}{56}$ ohms/metre/mm.², so that,

$$V = \frac{1{,}610}{55 \times 56} \text{ mm.}^2 = 0{\cdot}524 \text{ sq. mm.} = 0{\cdot}000812 \text{ sq. inch.}$$

4.—What is meant by " electrolytic corrosion " ? How may its destructive effects be kept under control ?

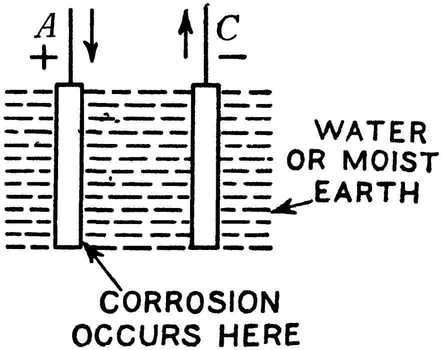


Fig. 3.

ANSWER.—When water is decomposed by electrolysis, oxygen is liberated at the anode and hydrogen travels with the current to the cathode. If the anode consists of a metal such as iron, copper, or lead which is attacked by the liberated oxygen, a salt of the metal forms and becomes dissolved in the water so that fresh metal becomes attacked until the whole of the anode is destroyed by this corrosion action (Fig. 3). In practice, the most important sources of such corrosion are d.c. traction systems in which the running rails form the return path for the current back to the power station. In this way a potential difference is produced between the distant end of the track and the earth so that the current leaks out from into the surrounding earth, and such leakage currents will spread in all directions when the ground is sufficiently moist. This is illustrated in Fig. 4 in which is shown a lead-sheathed cable buried in the earth and running parallel with the tram rail. The leakage currents from rail pass into the lead sheath and flow along it, eventually emerging to pass to the power station. Where the leakage current *leaves* the cable sheath forms the danger zone of corrosion.

One method for reducing the danger of such corrosion is to lay the cable in an iron conduit which is wrapped with tape and coated with bitumen. When the cables are laid in stone conduits, gaps can be arranged to interrupt the continuity of the current path in the cable sheath. Regulations made by the Ministry of Transport ensure that the dangers due to such underground corrosion shall be reduced to a minimum and one method of complying with these regulations is to install a " negative booster " in which an auxiliary D.C. machine (not shown in Fig. 4), is connected at its negative terminal to the distant end of the rail track and its positive terminal to the negative pole of the power station generator. In this way the current is sucked out of the rail to a sufficient extent to keep the amount by which the rail pressure is positive to earth down to a satisfactorily low value, the rail current is thus sucked back to the power station by an all-metal path. In order to keep the electric resistance of the rails as low as possible great attention must be paid to the bonding of the rails.

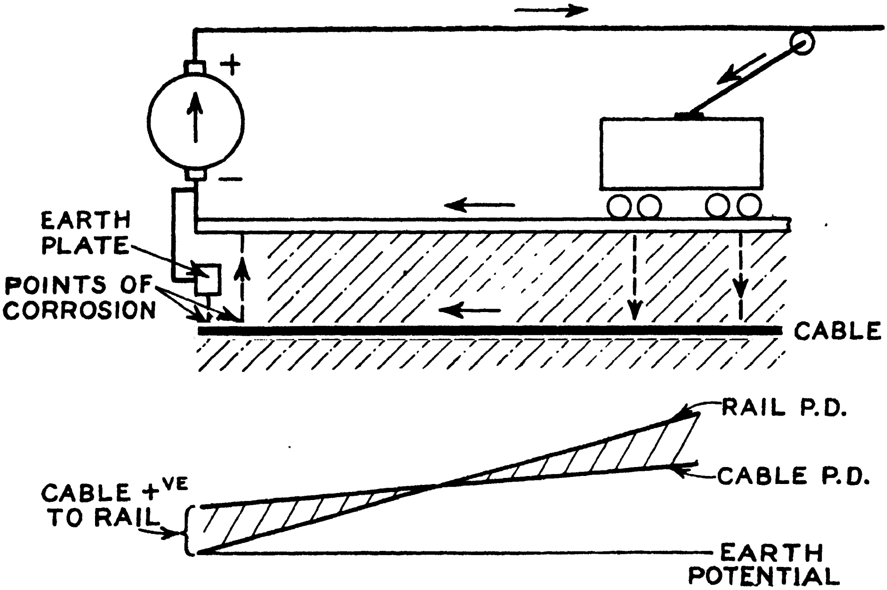


Fig. 4.

The question as to how great a pressure drop may be safely allowed along the rails is not easy to answer. For example, a pressure drop of from 2-5 volts gives rise at first to considerable earth leakage currents and consequently serious electrolysis effects. Soon, however, the effect of polarisation begins to become evident so that the available p.d. for electrolysis becomes greatly reduced. In any case, it is not considered desirable to allow a drop of pressure along the rails greater than from 5 to 7 volts.

5.—Experiment shows that when 1 gm. of hydrogen combines with oxygen to form water, 34 kcal. of heat energy are released. If, in the electrolysis of water, a current of I amperes flows for t seconds, find: (i) The necessary expenditure of energy. (ii) The minimum value of the applied p.d. for effective electrolysis. (iii) The energy required to release 1 cubic metre of hydrogen at 0° C. and 760 mm. pressure.

ANSWER.—(i) The electrochemical equivalent of hydrogen is $1{\cdot}044 \times 10^{-5}$ gm. per coulomb so that $I\,t$ coulombs will release

$$I \times t\,(1{\cdot}044 \times 10^{-5}) \text{ gm. of hydrogen at the cathode.}$$

The energy required to perform this release is $I \times t\,(1{\cdot}044 \times 10^{-5}) \times 34$ kcal., and since 1 kcal. $= 4{\cdot}18 \times 10^3$ joules, the energy required is,

$$I \times t\,(1{\cdot}044 \times 10^{-5})\,34 \times 4{\cdot}18 \times 10^3 = \underline{1{\cdot}47\,I \times t \text{ joules.}}$$

(ii) If V volts is the minimum pressure necessary for the electrolysis then the energy supplied is VIt joules, so that,

$$\underline{V = 1{\cdot}47 \text{ volts.}}$$

and this is the calculated value of the e.m.f. of polarisation shown in Fig. 20, Chapter II, page 53.

(iii) At 0° C. and a pressure of 760 mm. the density of hydrogen is 0·09 gm. per litre. Since 1 cubic metre $= 10^3$ litres, the mass of 1 c.metre of hydrogen is $0{\cdot}09 \times 10^3 = 90$ gm.

To release 1 gm. of hydrogen requires 96,540 coulombs, so that to release 1 c.metre requires $90 \times 96{,}540 = 8{,}688{,}600$ coulombs. Since, in practice, the necessary applied p.d. is about 1·7 volts, the energy which must be supplied to release 1 c.metre is

$1{\cdot}7 \times 0{\cdot}87 \times 10^7$ joules $= 1.48 \times 10^7$ joules, and since 1 kWh. is equal to $3{\cdot}6 \times 10^6$ joules, the required energy to release 1 c.metre of hydrogen is $\dfrac{14{\cdot}8 \times 10^6}{3{\cdot}6 \times 10^6} = \underline{4{\cdot}1 \text{ kWh. per cubic metre.}}$

Further, one volume of oxygen is released at the anode when two volumes of hydrogen are released at the cathode, so that the energy required for the electrolytic production of oxygen gas is $\underline{8{\cdot}2 \text{ kWh. per cubic metre.}}$

6.—What quantity of electricity will be necessary to produce one ton of caustic soda by the electrolysis of brine.

ANSWER.—Caustic soda (NaHO) is produced by the electrolysis of brine, that is, a strong solution of common salt (NaCl), the chemical change being defined by the equation,

$$\text{NaCl} = \text{Na} + \text{Cl}.$$

The sodium appears at the cathode and combines with the water to form caustic soda and hydrogen is in consequence released and appears at the cathode in accordance with the equation,

$$2\,\text{Na} + 2\,\text{H}_2\text{O} = 2\,\text{NaHO} + \text{H}_2$$

The electrochemical equivalent of soda is *0·0002386 gm. per coulomb* and one atom of soda (atomic weight *23·05*) *will* combine with one atom of hydrogen (atomic weight 1·008) and one atom of oxygen (atomic weight 16) so that 23·05 gm. of sodium will produce 40·058 gm. of NaHO. To release 23·05 gm. of sodium requires

$\frac{23}{0{\cdot}0002386}$ coulombs $= 9{\cdot}62 \times 10^4$ coulombs, and this will yield 40·06 gm. of NaHO. Hence,

1 ampere-hour = 3,500 coulombs will yield 1·5 gm. of NaHO, and the production of 1 ton of caustic soda, that is

$$\frac{2{,}240}{2{\cdot}2} \times 1{,}000 = 1{\cdot}02 \times 10^6 \text{ gm.}$$

will require $0{\cdot}68 \times 10^6$ ampere hours, or 680 amperes for 1,000 hours.

7.—If the electrochemical equivalents of iron, lead, and copper are respectively 0·000289, 0·0010714, 0·00066 gm. per coulomb, find what amount of each of these metals will become dissolved in electrolytic corrosion by a current of 1 ampere in 1 year, the number of active hours per annum being 4,200.

ANSWER.—*Iron :* 1 ampere for 4,200 hours = 4,200 × 3,600 coulombs and the weight of iron dissolved by this quantity of electricity is

$$0{\cdot}000289 \times 4{,}200 \times 3{,}600 \text{ gm.} = 4{\cdot}3 \text{ kgm.}$$

Similarly, the weight of *lead* dissolved will be 16·2 kgm., and the weight of *copper* dissolved will be 10 kg.

8.—Show how a logarithmic heating curve may be derived graphically from a knowledge of the heating time constant, one point on the curve and the final steady temperature.

ANSWER.—From the results given on pages 60 and 61, $\frac{d\theta}{dt} = \frac{\Theta - \theta}{Z}$ ....(i)

where Z is the heating time constant, Θ the final steady temperature, θ the temperature at any time t. When $\theta = 0$: $\frac{d\theta}{dt} = \frac{\Theta}{Z}$ and when $\theta = \Theta$: $\frac{d\theta}{dt} = 0$ so that the tangent drawn to the curve at 0 (Fig. 5) will give the intercept $A\,d = Z$ on the horizontal line through A. If a tangent to the curve be drawn at any other point T, the intercept $c\,b$ will also be equal to the time constant Z since $\left.\frac{d\theta}{dt}\right]_{\theta = T} = \tan\, c\,b\,T = \frac{\Theta - \theta_T}{c\,b}$ and a comparison of this equation with equation (i) shows that c b = Z.

This result leads to a very simple graphical construction for the heating and cooling curves when the time constant and any one point on the curve is known. In Fig. 6 draw the two verticals M M^1 and N N^1 distant Z apart and mark on N N^1 the final temperature Θ_1. Through the given point *Y* draw a line YP parallel to Θ_1 θ_Y, this line will then give the slope of the curve at Y so that after the small time interval △t, the temperature will .have risen to θ_X. Through the point *X* draw a line parallel to Θ_1 θ_X, then after the element of time △t the temperature will have risen to V. The procedure can then be continued to obtain the whole curve.

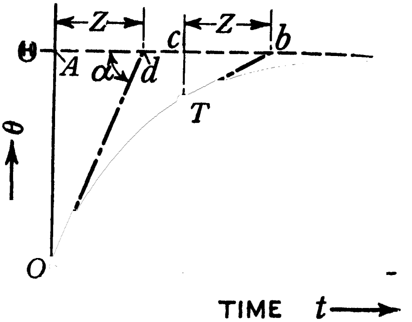


Fig. 5

If, when the temperature has risen to some such value as θ_A, the rate of heating is reduced so that the body begins to cool down, and if Θ_2 is the steady temperature towards which the curve tends to cool down, then a line drawn through A parallel to Θ_2 θ_A will give the slope at *A* of the cooling curve. After the time interval △t the temperature will have fallen to the value θ_B corresponding to the point B on the cooling curve, and so on.

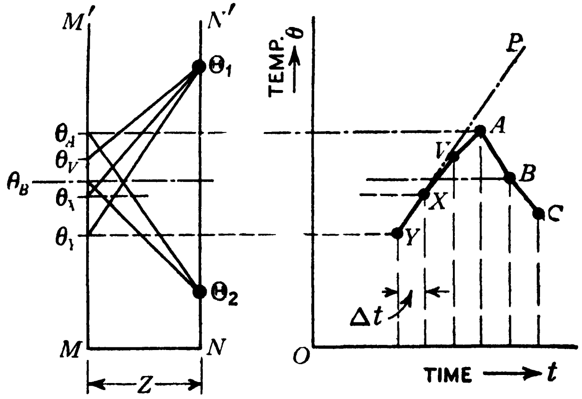


Fig. 6.

It will be seen that the curves are constructed by taking elementary portions of the tangents to the respective curves, and this construction is sufficiently accurate for most practical purposes, provided the magnitude of the time elements △t is not greater than 1/5 Z.

For practical purposes, the procedure may be made even more convenient if the heat supplied to the body is proportional to square of the load current, so that altering the current alters the final temperature in the ratio $\frac{\Theta_1}{\Theta_0} = \left(\frac{I_1}{I_0}\right)^2$, where Θ_0 is the permissible final steady temperature when the load current is I_0, and Θ_1 is the final steady temperature to which the curve tends when the current is altered to the value I_i. Then, in Fig. 7, the verticals M M^1 and N N^1 are drawn at the distance Z apart, and the current is marked on N N^1 and the corresponding temperature is marked on M M^1. If, for example, the current is reduced to $\frac{I_0}{2}$, the corresponding final temperature Θ_1, Fig. 7, will be such that the height $M\,\Theta_1 = \left(\frac{\frac{1}{2}\,I_0}{I_0}\right)^2 M\,\Theta_0 = \frac{1}{4}\,(M\,\Theta_0)$. In this way, the heating and cooling curves can easily be plotted when the corresponding changes of the load current are known.

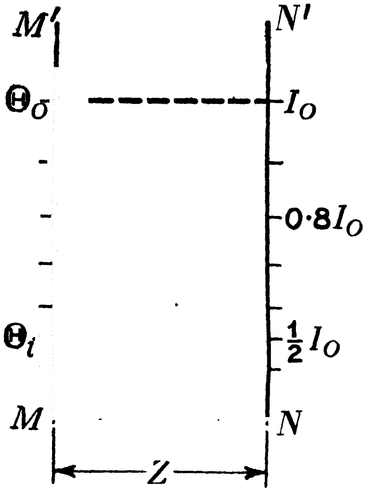


Fig. 7

9.—The field coil of a generator has a heating time constant of 20 minutes and heat is generated at a uniform rate such that the final steady temperature will be 80° C. If the room temperature is 20° C. construct the temperature/time curve for the first 20 minutes after switching on the coil when it is at room temperature. If the current is then switched off, construct the cooling curve for the next 10 minutes.

ANSWER.—In Fig. 8 are shown the required heating and cooling curves, and these have been drawn exactly in accordance with the procedure described in the solution to Question 8. It is to be noted here that since the datum temperature is now the room temperature of 20° C. the temperature Θ in Fig. 8 is the temperature *rise* above 20° C.

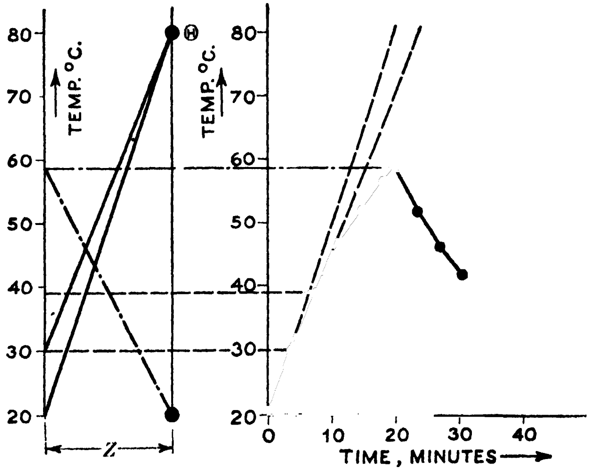
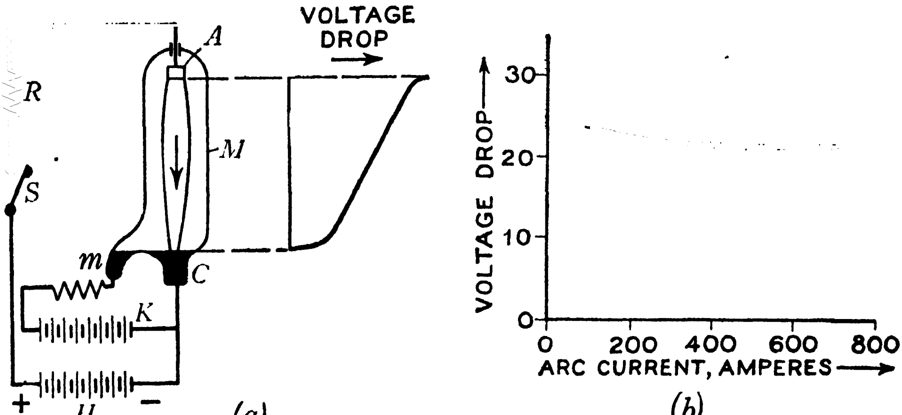


Fig. 8.

10.—*Explain why a mercury-arc rectifier is not very suitable for operating at low pressures.*

Answer.—In Fig. 9*a* is shown diagrammatically a sealed glass bulb *M* with the anode *A* (the active surface of which is usually graphite) and a mercury pool cathode *C*. A battery *H* is connected through a switch *S* and a resistance *R*, between the anode and cathode, the path of the mercury vapour arc being shown by the arrow. An auxiliary pool of mercury *m* is shown in series with a resistance and a battery *K* and is used for ignition as follows. By tilting the bulb *M* so that the pool *m* overflows into *C*, the mercury bridges the gap, the circuit of the battery *K* becomes closed, and a current flows through the mercury bridge. If the bulb *M* is now restored to its vertical position, the mercury bridge becomes ruptured and so gives rise to an arc and thus the hot cathode spot on the surface of the pool *C* is produced. If the switch *S* is now closed, current from the battery *H* will pass through the bulb *M* as shown by the arrow. The auxiliary ignition battery *K* is now switched off and the current through the bulb is maintained by the battery *H*. In Fig. 9*b* is shown the relationship between the total pressure drop in the bulb and the current. The curve shows that the pressure drop is practically constant for all values of the current, a consequence of this being that the efficiency of the rectifier arc is practically constant and independent of the load. This characteristic of the mercury-arc rectifier is one of its very valuable features.



(*a*) (*b*)

Fig. 9.

If V is the supply pressure, I is the load current and v is the pressure drop in the rectifier, this being taken to be a constant quantity for all load currents, then:

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}} = \frac{VI - vI}{VI} = \frac{V - v}{V}$$

and is consequently independent of the current I. This expression also shows that, since the total pressure drop v is approximately the same for high values of the operating pressure as for low values, the efficiency will increase as the operating pressure increases as is evident from the expression

$$\text{Efficiency} = \frac{I - v/V}{I}$$

11.—Describe the principal features of a single-phase mercury-arc rectifier.

Answer.—In Fig. 10 are shown diagrammatically the essential features of the circuit connections for a single-phase mercury-arc rectifier The primary winding of the transformer T is connected to the source of alternating current supply pressure and the secondary winding is provided with a centre tapping point which is connected to the terminal X, thus forming the negative terminal of the direct current supply. Two anodes A_1 and A_2 are shown sealed into the glass bulb and the mercury pool cathode C is connected to Y, which thus forms the positive terminal of the d.c. supply.

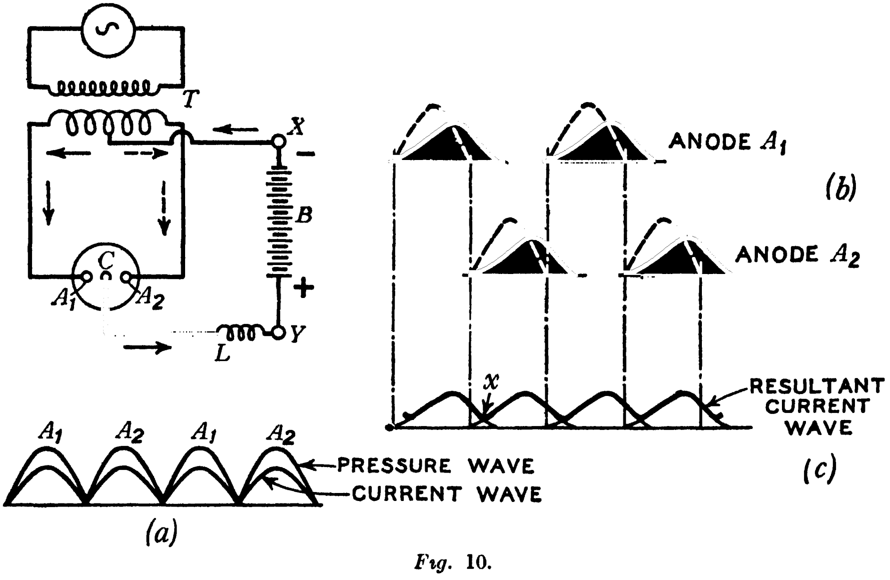


Fig. 10.

In Fig. 10 the rectifier is shown as used for charging a battery B, an inductance L being shown in series with the d.c. circuit, and the function of this inductance will be explained later. The auxiliary ignition circuit has not been included in Fig. 10. A little consideration will show that, during one half-wave of a cycle of the a.c. supply pressure, the anode A_1 becomes positive with respect to the cathode C and during the other half-wave the anode A_2 will be positive to the cathode C, so that each half-cycle will in this way produce a current in the same direction through the d.c. circuit. That is to say, a uni-directional, but heavily pulsating current, will be produced in the d.c. circuit.

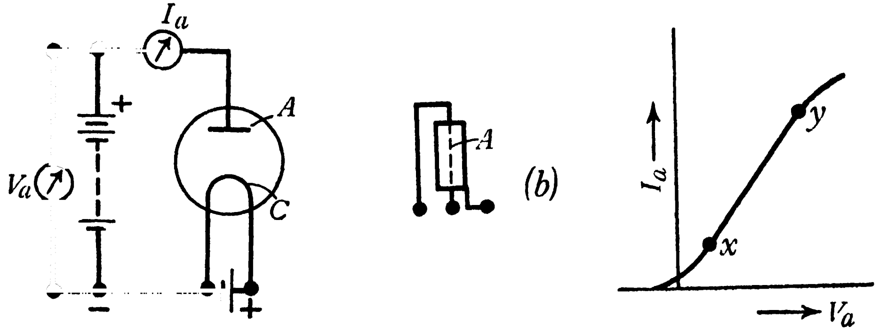
When the inductance L is omitted from the circuit, the rectifier current wave in the d.c. circuit will be as shown in Fig. 10*a*, from which it will be seen that the current falls to zero at the end of each half-cycle, simultaneously with the pressure. Now it is a characteristic feature of the mercury-arc that if the current ceases even for one hundred-thousandth part of a second, the arc will be extinguished and the rectifier will consequently cease to function. One means for eliminating this difficulty is to include the inductance L in the d.c. circuit, the result of this being that when the applied pressure wave falls to zero, the current will not immediately cease to flow, because the electromagnetic energy which is stored in the inductance will sustain it, so that the current waves are prolonged as shown in Fig. 10*b*. That is to say, the cathode hot spot persists whilst the arc passes over from one anode to the other.

The current in the d.c. circuit is obtained by combining the two sets of current waves shown in Fig. 10*b* and their resultant current wave is then shown by the heavy line curves of Fig. 10*c*. It will be seen that at the points such as x where the current waves cross, the resultant current is not given by the sum of the two corresponding ordinates. Actually, the anode A_1 will carry the current up to the point x and the arc will then pass over to the anode A_2. That is to say, the two anodes will not be in operation simultaneously, although a method by which two anodes may be caused to be in action simultaneously, is now being used in practice for large rectifiers.

It is seen from Fig. 10*c* that although the current in the d.c. circuit is still strongly pulsating, the fluctuation above and below the mean value of the uni-directional current is much less than for the conditions to which Fig. 10*a* applies. A direct current which pulsates to the extent shown in Fig. 10*a* would be of restricted value, except for some special applications such as battery charging. For the operation of electric motors, however, it is necessary to eliminate the fluctuations to a very much greater degree, and this can be done by the use of a rectifier bulb built to operate from a three-phase supply system.

12.—Explain the operation of the two-electrode thermionic valve (the diode).

Answer.—In Fig. 11*a* are shown the essential features of the diode valve, comprising an evacuated glass bulb containing a tungsten filament or cathode *C* opposite to which is the anode plate *A*. Another method of arranging the anode plate is shown in Fig. 11*b* in which the filament is set along the axis of a cylindrical form of anode. The filament is heated by a current from a cell and a battery of cells is connected across the anode plate and one terminal of the filament. When heated to the appropriate temperature, the filament ejects electrons and these, in the absence of an anode pressure, would gather like a cloud round the filament as a " space charge," thus obstructing further electron emission from the filament. When a positive potential is applied to the anode plate, electrons are attracted by the plate so that a continuous stream of electrons will flow from the filament into the plate and thence, through the battery back to the filament circuit. In the normal technical usage, a stream of *electrons* (negative electricity) passing from the *filament* to the *anode* plate corresponds to a current of *positive* electricity passing from the *anode* to the *filament*.



(*a*) *Fig.* 11. (*c*)

If the relationship between the anode pressure and the anode current is measured by instruments connected as shown in Fig. 11*a*, a graph such as that of Fig. 11*c* will be obtained, of which the portion *x y* is practically a straight line. When the anode pressure is negative, the anode current becomes vanishingly small. When the anode pressure exceeds a certain positive value the anode current tends to reach a limiting value which corresponds to the condition of *saturation*, that is, all the electrons emitted from the filament will be drawn into the anode plate at the same rate at which they are being emitted, so that no further increase of current can then take place by increasing the anode voltage.

The characteristic of Fig. 11*c* shows that the diode valve has a *rectifying property* and it is consequently used for the detection of radio waves as well as for converting alternating current into direct current.

13.—Show how a three-electrode thermionic valve (a triode) may be used as an amplifier.

Answer.—For this purpose the three-electrode valve can be connected as shown in Fig. 12. In between the anode plate and the cathode filament the third electrode, viz., the " grid," is sealed into the glass bulb. This electrode is a perforated plate such as a strip of wire mesh or a zig-zag arrangement of wire, and is connected to a battery by means of an adjustable potential slide resistance so that the grid is maintained at a negative pressure with respect to the filament, the value of this negative pressure being termed the " grid bias." Across the anode and the filament is connected a battery in series with a resistance R_A. If the relationship between the grid pressure and the anode current is plotted, a curve such as that shown in Fig. 13 is obtained.

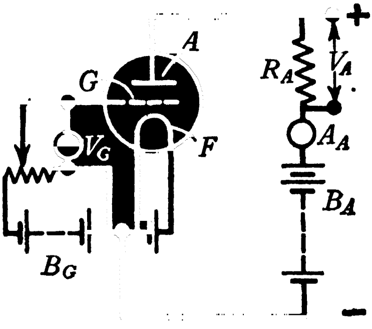


Fig. 12

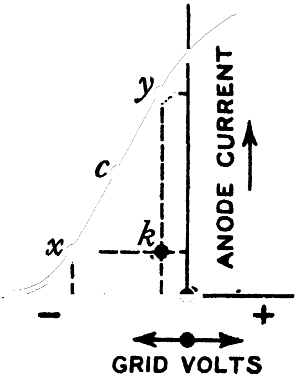


Fig. 13

A range of the curve, e.g., x to y, is practically a straight line, so that if the pressure which is applied to the grid is varied within this range the p.d. across the anode resistance will vary proportionally, that is, if the grid bias is set to correspond about to the mid-point c of the straight line portion of the characteristic of Fig. 13, and if an a.c. voltage be applied across the grid so that its amplitude remains within the range xy of the characteristic, then the voltage which will appear across the anode resistance will vary exactly proportionally, and with an increased amplitude. In Fig. 13 the ratio

$$\frac{yk}{xk} = \frac{\text{change of anode current}}{\text{change of grid pressure}}$$

is termed the mutual conductance g_m of the valve, and, for brevity, is sometimes referred to as the " slope of the valve."

In the normal types of valves as used for post-office repeaters the ratio

$$\frac{\text{impressed a.c. voltage on the grid}}{\text{a.c. voltage across the anode resistance}} = \text{about } \frac{1}{10}$$

and, consequently, the amplification factor is 10. A higher degree of *amplification* can be obtained by substituting a " step-up " transformer for the anode resistance R_A.

14.—*How may the cathode-glow discharge be applied as a surge arrester ?*

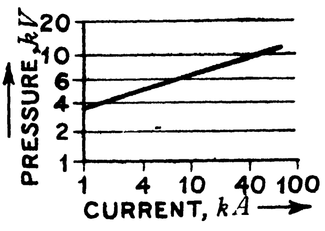


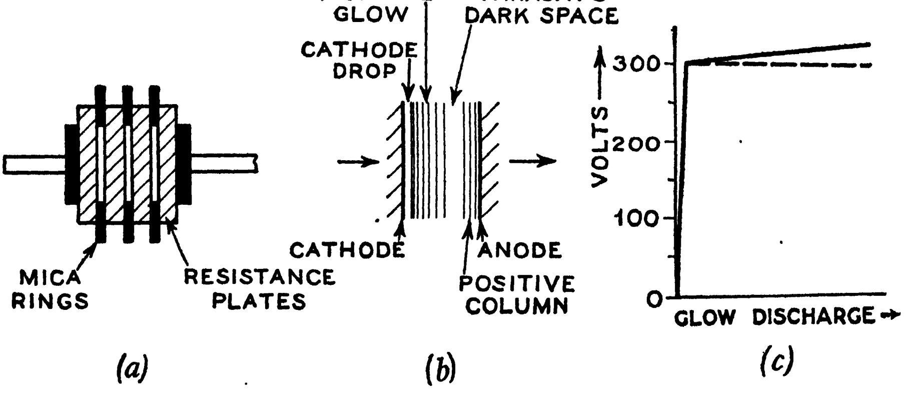
Fig. 14.

Answer.—It is an experimental fact that if the gap between two disc electrodes across which a p.d. exists is reduced say from a few mm., the breakdown voltage at atmospheric pressure of the air-space between them will be reduced until a minimum value of about 350 volts is reached, even when the gap is reduced to about 0.01 mm.

If the discharge between two electrodes is stabilized when the gap width has this critical value, it is found that a p.d. of about 300 volts is necessary to maintain the glow-discharge. When the striking pressure is 350 volts, the pressure will fall to 300 volts when the discharge is established, and when the pressure falls below this value the discharge will cease.

In order to stabilize the discharge, a special resistance material is used formed of semi-conducting crystals with a binding mixture containing a small amount of conducting substance, and the volt-ampere characteristic of such a synthetic material is shown in Fig. 14 (plotted to a logarithmic scale). It will be seen from this graph that for an increase of current from 1 kA. to 50 kA. the pressure rises only from 4 kV. to about 10 kV.

In Fig. 15*a* is shown a part of the assembly of discharge gaps as applied to a surge arrester, and Fig. 15*b* shows the glow-discharge between the



(a) (b) (c)

Fig. 15.

electrodes. In practice, a spark-gap is connected in series with the arrester, and the breakdown of this spark-gap initiates the glow-discharge. Fig. 15c gives the pressure characteristic of the discharge showing the effect of the critical gap pressure of 300 volts. Recent experimental results have shown that this type of surge-arrester can carry a current of 300,000 amperes without being destroyed, and there is good reason to believe that it will be capable of protecting a transmission line against direct lightning strokes into the line, thus eliminating the most potent menace to the continuity of high tension power transmission systems.

15.—Give an account of Townsend's "avalanche" theory of the breakdown of a spark-gap.

Answer.—The mechanism of the electric spark discharge has been the subject of intensive research for many years and the problem has proved to be an extremely difficult one to unravel since it involves the simultaneous operation of many interactions of electrical particles which contribute to the transformation of the initially insulating gas into a medium of relatively high conductivity. J. S. Townsend gave a simple account of the "avalanche" process by which a spark-discharge is evolved, and this is recognized in principle as a true statement of the facts. It was, however, soon realized that this description was not complete since, for example, it does not cover the case of a spark-discharge in a gas at atmospheric pressure and such a case is of outstanding importance from a technical point of view.

The following is a brief description of Townsend's theory. Suppose a plate condenser has air at a low pressure as the dielectric, and assume that there is applied to the plates a p.d. of slightly greater magnitude than the breakdown pressure. If, now, an electron is released from the negatively charged plate, say by irradiation by an ultra-violet ray, this electron will begin to travel towards the positively charged plate, and on its journey will ionize the air by impact and, together with the secondary electrons so released, will give rise to an "avalanche-like" growth of the total number of released electrons since each one will, by impact, release another group and so on. The initial process of formation of this avalanche can be made visible by means of Wilson's "cloud-chamber" and its characteristics (speed, diameter, etc.) can be determined in this way. In Fig. 16 is shown such an avalanche growth and this diagram has been drawn from a photograph which was taken by means of the cloud-chamber. The principle of operation of this experimental appliance is this: When a dust-free gas is saturated with water vapour, there will be produced, at a certain stage of cooling—and this can be obtained by a rapid expansion of the gas—a

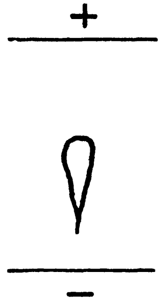


Fig. 16.

condensation of the water vapour on the positive and negative ions and this condensation takes place more readily on negative than on positive ions. In this way the path of the impact ionizing electrons become visible by condensation of the water vapour (i.e., the cloud) on the ions so formed and the effect can be photographed.

The foregoing description of Townsend's theory of the mechanism of a spark-discharge leads to results which are in good agreement with actual observations when in the region of low gas pressures, that is, up to about 100 tor* with 1 cm. distance between the electrodes. At higher values of the gas pressure—or more correctly expressed, for values greater than 1,000 tor-cms. (pressure in tors × gap electrode in cms.), cloud-chamber observations show that at a certain critical increase of the electric field intensity, the initially formed avalanche changes during its travel, to a channel-form of discharge. One consequence of this extended account is that it explains the much shorter building-up time required at high gas pressures than the Townsend theory can account for, and it makes clear what means can be provided to control the speed of discharge and thus achieves a result which is of very great technical importance.

16.—Give an account of the principal phenomena associated with an electric arc.

ANSWER.—A typical arc characteristic is shown in Fig. 17, in which v_{ig}. denotes the pressure which is necessary to start the arc across a gap. The current then increases and the arc pressure falls in accordance with the curve *A B*. If the current is reduced, the arc pressure rises according to the curve *B C*, so that when the arc becomes extinguished the pressure across the gap is v_{qu}, the " quenching pressure." This exceptional relationship between the pressure and current of an arc requires a special

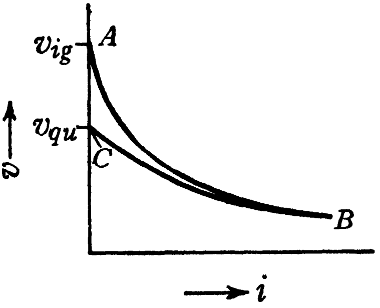


Fig. 17.

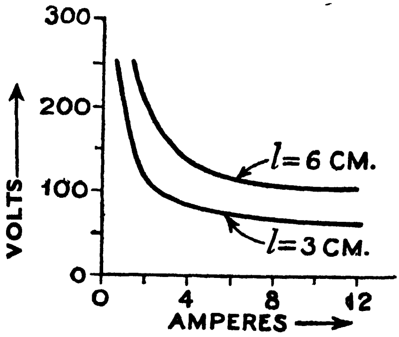


Fig. 18.

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* The pressure of a column of mercury 1 millimetre high is 1 tor.
A column of mercury at 0°C. of 1 sq. cm. cross section and 76 cm. high is 1.03 kg./cm^2.
1 Physical atmosphere = 1 Atm. = 760 tor.
1 Technical atmosphere = 1 at = 1 kg./cm^2.

treatment of the circuit performance in all cases in which an arc is comprised *in the circuit, and it is of paramount importance for the study of circuit*-breaker and fuse performance.

The relationship between pressure and current for a steadily burning arc can be taken as an average between the two branches shown in Fig. 17, and two such characteristics are shown in Fig. 18 for arcs of different lengths, between copper electrodes. This relationship may be expressed by the equation

$$V = a + \frac{b}{I}$$

from which it is seen that the pressure falls as the current increases in accordance with a hyperbolic law. The constants a and b depend upon the length l of the arc, viz.,

$$a = \alpha + \gamma\, l$$
$$b = \beta + \delta\, l.$$

For an arc between copper electrodes Rüdenberg gives

$\alpha = 30$ volts : $\gamma = 10$ volts/cm.

$\beta = 10$ volt-amperes : $\delta = 30$ volt-amperes/cm.

The general nature of the arc characteristic may be accounted for as follows : The pressure drop between the arc electrodes has the character shown in Fig. 19. That is to say, the total pressure drop comprises the anode drop v_A, the cathode drop v_c, and the drop v_l in the arc column. The anode drop and the cathode drop are each dependent to some degree upon the current strength, whilst the drop in the arc column is very nearly proportional to the length of the column.

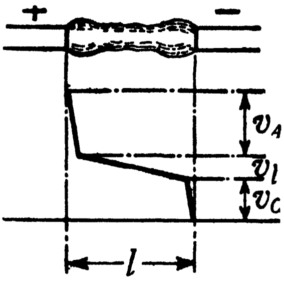


Fig. 19.

From the incandescent tip of the cathode, electrons stream out under the action of the electric field between the electrodes. The electrons strike the neutral molecules in their path and ionize them and the component parts of the ionized molecules then move towards the respective electrodes under the action of the electric field and bombard them ; viz., the negative particles move at a high speed towards the anode and the positive components, being much heavier, move at a correspondingly slower speed towards the cathode. This bombardment gives rise to incandescent areas on the respective electrodes, and if this sequence of events reaches a stable condition, the arc persists, if not, the arc will be quenched. It follows, therefore, that it is not possible to maintain an arc having any given values of current, pressure, and length, but that there is a definite relationship between these quantities which is termed the " arc characteristic." The movement of the electric particles to the respective electrodes results in a massing of electrons near the anode and of the positive ions near the cathode, and it is this concentration of the particle which gives rise to the respective drops at the electrodes. About 80 per cent. of the current in the arc is accounted for by the travel of the electrons.

The temperature of the incandescent spot on the cathode for a copper arc is from 2,000° C. to 3,000° C. and for a carbon arc is from 3,000° C. to 4,000° C., it being observed that since copper is a very good conductor of heat, the temperature of the incandescent spot is cooled by the metal in its neighbourhood, whilst carbon, being a relatively poor conductor of heat, the cooling effect on the incandescent spot is very much less. The temperature of the gas in the arc is from 8,000° C. to 10,000° C., the lower value being obtained for small values of the current and the higher value for greater current strengths.

The current density at the cathode spot is generally some thousands of amperes per sq. cm., whilst for carbon arcs it is some hundreds of amperes per sq. cm. The carbon arc produces the highest temperatures which so far it has been found possible to produce by any known technical means, and the energy content of the arc per unit volume, that is, the " specific energy," is considerably higher than that which can be obtained in any other known type of furnace.

For a carbon arc the cathode pressure drop v_c is about 10 volts and occurs over a length of about 10^{-4} cm.; that is, there is a pressure gradient of about 10^5 volts per cm. and the current density at the cathode is about 500 amperes per sq. cm. The positive ions near the cathode therefore acquire a speed which corresponds to this enormous pressure gradient and, striking the cathode with great energy, raises the temperature to the boiling point. This impact of ions on the cathode produces an emission of electrons which move away at a very high speed under the high pressure gradient in the neighbourhood. The magnitude of the electron emission is governed by the nature of the material of the cathode and its chemical composition. If the cathode consists in part of metallic oxide for which the energy of emission is small, the cathode drop will be correspondingly small.

17.—*What is the International Electrotechnical Commission (I.E.C.) definition for the specific resistance of standard soft copper? From this definition derive the temperature coefficient at 0° C.*

ANSWER.—The specific resistance of standard soft copper is defined as follows:

At a temperature of 20° C. a soft copper wire 1 metre long and of uniform cross section of 1 sq. mm. has a resistance of 1/58th ohm, that is 0·01724 ohm.

The density at 20° C. is 8·89 gm. and the temperature coefficient is $\frac{1}{254{\cdot}5}$ that is, 0·00393 per 1° C.

It follows from this definition that the resistance at 20° C. of a standard soft copper wire of uniform cross section, of length 1 metre and mass 1 gm., is $(0{\cdot}01724 \times 8{\cdot}89) = 0{\cdot}1533$ ohm. (See British Standard Specification No. 125.)

The expression relating the specific resistance and the temperature is

$$\rho_{\Delta t} = \rho_{20}(1 + \alpha t)$$

that is

$$\rho_{\Delta t} = \frac{1}{58}(1 + 0{\cdot}00393\Delta t)$$

where Δt is the temperature rise above 20° C., so that

$$R_{\Delta t} = R_{20}(1 + 0{\cdot}00393\Delta t)$$

At 0° C.: $\Delta t = -20°$ C., so that

$$\rho_0 = \frac{1}{58}(1 - 0{\cdot}00393 \times 20) = \frac{1}{58} \times 0{\cdot}9124 = \frac{1}{62{\cdot}9}$$

Hence, the temperature coefficient at 0° C. is given by the equation

$$\rho_{20} = \rho_0\left\{1 + (\alpha \times 20)\right\}: \text{ or } \frac{1}{58} = \frac{1}{62{\cdot}9}(1 + 20\,\alpha)$$

from which it follows that the temperature coefficient at 0° C. is

$$\alpha = 0{\cdot}0042 \text{ per } 1° \text{ C.}$$

18.—Derive an expression which shows how the temperature rise of a coil which carries a current of I amperes depends upon the way in which the temperature coefficient of the material of the wire varies with the temperature. It is to be noted that the cooling is assumed to be forced so that the radiated heat is negligibly small as compared with the heat which is dissipated by convection.

ANSWER.—When the temperature has reached a steady final value the heat energy supplied will be equal to the heat energy dissipated, so that

$$W = \beta_0\Delta t = I^2R_a(1 + \alpha \times \Delta t)$$

and the final temperature rise will be

$$\Delta t = \frac{1}{\dfrac{\beta_0}{I^2R_a} - \alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

where β_0 is the heat dissipated per 1°C. temperature rise (that is, the {specific cooling constant × cooling surface}),

I^2R_a is the heat energy generated per second for a resistance R_a at 15°C.

α is the temperature coefficient at 15°C., viz., $\alpha = 0{\cdot}004$ for copper and $0{\cdot}00384$ for aluminium.

(i) In order to give a final temperature rise (for example) of 100°C. let the current be $I \doteq 1$ unit. If the temperature coefficient α is neglected then

$$\Delta t = I^2 \frac{R_a}{\beta_o}$$

that is, the temperature will be directly proportional to the square of the current so that, for example, if the current is increased to $I = 1{\cdot}2$ units, the temperature rise will be 1·44 units or 144° C. In Fig. 20, Curve 1 shows the relationship between current increase and final temperature obtained in this way.

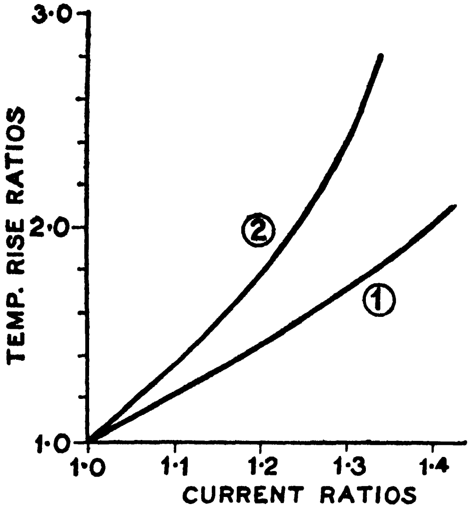


Fig. 20.

(ii) When the temperature coefficient α is taken into account, the conditions are widely different. The steady temperature will then be reached when

$$\frac{\beta_o}{R_a} = I^2\left(\frac{1 + \alpha \times \Delta t}{\Delta t}\right)$$

so that for $I = 1 : \Delta t = 100 : \frac{\beta_o}{R_a} = 0{\cdot}14$, and consequently the temperature rise for any other current will be

$$\Delta t = \frac{1}{\frac{0{\cdot}014}{I^2} - \alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

that is, if $I=1.2$, then $\Delta t = 1{\cdot}7$ instead of 1·44 as found in the previous case when the temperature coefficient was neglected. The curve 2 in Fig. 20 shows the relationship between temperature rise and current as derived in this way. For example, the equation (ii) shows that when $\frac{0{\cdot}014}{I^2} = \alpha$, that is, when $I^2 = 3{\cdot}5 : I = 1{\cdot}7$ and the temperature rise is infinitely great.

19.—Describe some modern form of the carbon microphone.

ANSWER.—The carbon contact type of microphone was first applied as a telephone transmitter by Prof. Hughes in America, and comprised a carbon rod supported at each end by a recessed carbon block. For many years this device was the basic principle for further microphone development.

In the form used by Hughes, however, it was subject to many disadvantageous features, and in its original form it was little more than a laboratory appliance. The point-like contacts of the carbon rod were not stable in operation, and the least irregularity in its setting, or the least particle of foreign matter in the contacts, was sufficient to put it out of action. It was also extremely sensitive to noise in its vicinity, and this could greatly obscure the intelligibility of the speech sounds. In consequence of these defects efforts were directed to the use of multiple contacts, and a modern type of such an arrangement is shown in Fig. 21.

A base of nickel-plated brass supports an insulated plate of carbon which forms one electrode. In the upper surface of this carbon plate a trough-like recess is formed to hold carbon granules. These granules are about 1 mm. in diameter and are held in the recess by a carbon diaphragm which is fixed slightly in front of the carbon plate and forms the other electrode. A perforated protecting disc is sprung into position in a groove in the rim of the brass supporting frame and so holds the carbon diaphragm at its rim. The resistance of the microphone when not in operation is from 10 to 40 ohms.

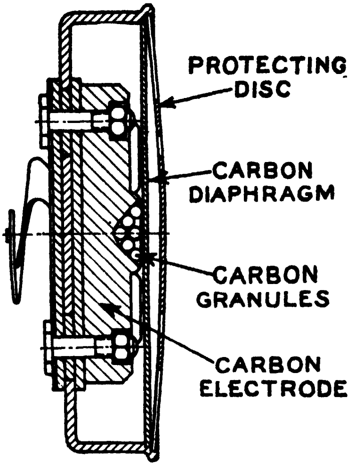


Fig. 21

20.—*A source of D.C. supply of which the e.m.f. is E volts and the resistance is R_0 ohms is connected to a consumer's load of resistance R_x ohms. Find (i), the conditions for which the maximum power is delivered to the consumer, and (ii), the efficiency of the supply system under these conditions.*

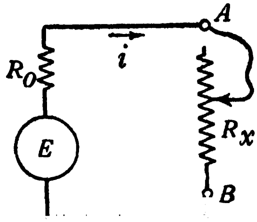


Fig. 22

ANSWER.—The circuit diagram for the given conditions is shown in Fig. 22.

(i) The power supplied to the consumer's terminal *AB* is

$$i^2R_x \text{ watts}$$

where

$$i = \frac{E}{R_0+R_x} \text{ amperes.}$$

Hence the consumer's power may be expressed as

$$W_x = \frac{E^2}{(R_0+R_x)^2} R_x \text{ watts.}$$

and this quantity will be a maximum when

$$\frac{d}{dR_x}\left[\frac{E^2R_x}{(R_o+R_x)^2}\right]=O$$

from which it is found that for W_x a maximum

$$\underline{R_x = R_o}$$

that is, when the internal resistance of the source of current supply is equal to the consumer's load.

(ii) The efficiency of the supply for the condition $R_x=R$ is

$$\eta\ \% = \frac{\text{Output power}}{\text{Total power consumed}}\ 100 = \frac{i^2R_x}{i^2(R_o+R_x)}\ 100$$

and, inserting $R_x=R_o$ gives

$$\left.\eta\,\%\right] = 50\%$$

for maximum power to the consumer.

Chapter III

COULOMB'S LAW : FIELDS OF ELECTRIC FORCE : POTENTIAL

1.—When the insulation between an overhead transmission line and the supporting mast fails, a current will flow to earth and, for many purposes, a sufficiently good approximation can be obtained by assuming that the current passes into the earth through a hemispheric surface bedded in the earth and representing the foundation of the mast. Obtain expressions for the resultant current density and the pressure drop per centimetre at points in the neighbourhood of the base of the mast and relate these quantities to corresponding quantities for the electric field due to a charged sphere.

ANSWER.—Fig. 1 shows the equivalent hemispherical surface of diameter D cms. through which the current from the live mast is assumed to enter the earth, the total earth current which thus passes through this electrode being I_e amperes. Fig. 2 shows an insulated charged sphere of diameter D cms., the charge on the sphere being $2Q$ electrostatic units so that the charge on one hemisphere is Q electrostatic units.

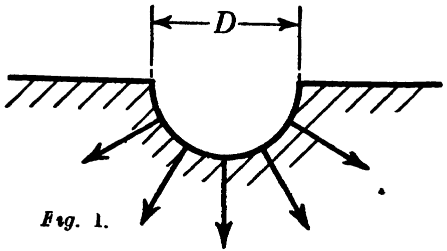
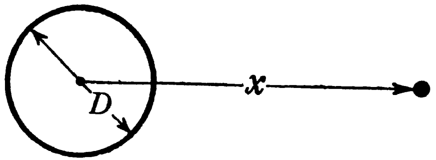
It may then be said that the total earth current I_e which is the source of the current density i_x in the earth at a point distant x cm. from the centre of the hemispherical electrode, *corresponds* to the charge Q on the hemisphere which is the source of the density of electric charge σ_x at a distance x cm. from the centre of the sphere.

In the accompanying Table, a series of such corresponding quantities is given and these provide a helpful means for solving a number of problems relating to earth currents to which overhead transmission lines give rise. It is to be noted, for example, that the " diffusion resistance " R_u ohms as defined in item (v) of the Table is such that when the potential V_1 volts of the live mast with respect to earth is known, then the ratio

$$\frac{V_1}{R_u} = I_e \text{ amperes,}$$

and is the total earth current which flows to earth from the mast through the hemispherical electrode as shown in Fig. 1. By relating this diffusion resistance to the reciprocal of the capacitance of a similar electrode, as shown in the Table, the total current flowing to earth through various different types of electrodes can be written down at once since the corresponding capacitance is known from standard formulae. The practical significance of this procedure will be seen from the numeric data given in Example 4 (See also T. F. W. " Engineering," p. 478, May 10, 1940).

TABLE OF QUANTITIES AND FORMULÆ.

 Fig. 1.	Corresponding Quantities	 Fig. 2
(i) TOTAL CURRENT through Surface of Hemisphere $= I_e$ Amperes	$I_e \equiv Q$	(i) CHARGE ON HEMISPHERE $= Q$ Electrostatic Units.
(ii) CURRENT DENSITY At Surface of Hemisphere : $i_0 = \frac{I_e}{2\pi\left(\frac{D}{2}\right)^2} = \frac{2 I_e}{\pi D^2}$ Amps./cm^2 At Distance x cms. : $i_x = \frac{2 I_e}{\pi (2x)^2} = \frac{I_e}{2\pi x^2}$ Amps./cm^2	$i \equiv \sigma$	(ii) DENSITY OF ELECTRIC CHARGE At Surface of Sphere $\sigma_0 = \frac{2Q}{\pi D^2}$ Elec.-st. Units/Cm^2 At Distance x cms. from Centre of Sphere : $\sigma_x = \frac{Q}{2\pi x^2}$
(iii) PRESSURE DROP $v_x = i_x s$ Volts/Cm. where s is Specific Resistance of Earth in Ω/cm./cm^2	$v \equiv f$ $s \equiv \frac{4\pi}{\varepsilon}$	(iii) ELECTRIC FORCE (a) At Surface of Sphere : $f_0 = \frac{4\pi\sigma_0}{\varepsilon}$ At Distance x cms. from Centre : $f_x = \frac{4\pi\sigma_x}{\varepsilon}$ El.-st. Units/Cm.
(iv) TOTAL PRESSURE RISE at Electrode Surface $V_I = \int_\infty^0 v_x\,dx = \frac{s\,I_e}{\pi D}$ Volts	$V_I \equiv V_0$	(iv) POTENTIAL OF SPHERE $V_0 = \int_\infty^{D/2}\left(f_x\,dx\right) = \frac{4Q}{\varepsilon D}$ El.-st. Units.
(v) $\frac{V_I}{I_e} = \frac{s}{\pi D} = R_u\,\Omega$: the "Diffusion Resistance" of the Earth.	$R_u \equiv \frac{1}{\left\{\text{Capacitance of the Field Due to Charged Hemisphere.}\right\}}$	(v) $\frac{V_0}{2Q} = \frac{1}{\text{Capacitance of Sphere}} = \frac{2}{\varepsilon D}$ $\therefore \frac{\varepsilon D}{4} = \left\{\text{Capacitance of Field Due to the Charged Hemisphere.}\right.$

2.—Assuming the current from a live mast radiates from a hemispherical surface bedded in the earth, find the potential and the current density at a point distant x cm. from the centre of the hemisphere.

Answer.—From the Table (see solution to Question 1), the current density at a point distant x cm. from the centre of the hemispherical electrode is,

$$i_x = \frac{I_e}{2\pi x^2} \text{ amperes/cm.}^2$$

so that the pressure *drop* in volts per cm. at this point will be

$$v_x = si_x = \frac{sI_e}{2\pi x^2} \text{ volts/cm.}$$

where s is the specific resistance of the earth in ohms/cm./cm.

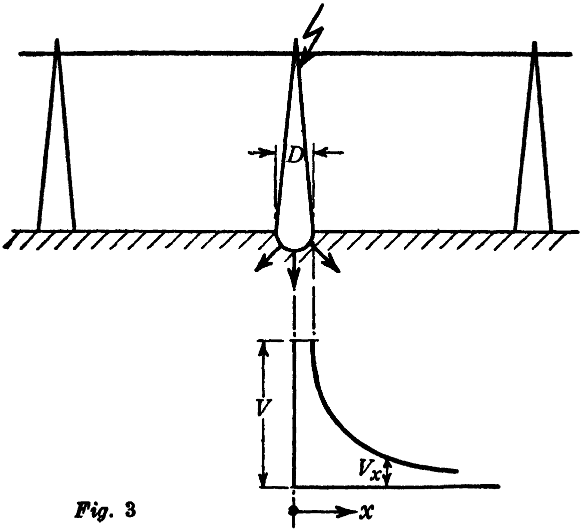


Fig. 3

The pressure *rise* at any point distant x cm. from the centre of the hemisphere will be

$$V_x = -\int_{\infty}^{x} v_x dx = \frac{sI_e}{2\pi x} \text{ volts}$$

and this relationship between V_x and x is shown in Fig. 3.

The diffusion resistance at a point distant x cm. from the centre of the hemisphere is

$$(R_u)_x = \frac{V_x}{I_e} = \frac{s}{2\pi x} \text{ ohms}$$

and at the surface of the hemispherical electrode is $R_u = \frac{s}{\pi D}$ ohms.

3.—An animal is standing in the neighbourhood of a live mast. Find the magnitude of the p.d. which will be short-circuited by the body of the animal.

ANSWER.—In Fig. 4 the span of the animal is shown as *G* cms., so that from the results obtained in Example 2 the p.d. which is short-circuited by the animal is

$$V_G = -\int_x^{x+G} v_x dx = -\frac{sI_e}{2\pi}\int_x^{x+G}\frac{dx}{x^2} = \frac{sI_eG}{2\pi x(x+G)} \text{ volts}$$

(i) If x is large as compared with G, then $V_G \simeq \frac{sI_eG}{2\pi x^2}$ volts

(ii) If the animal is standing near the live mast, then

$$x = \frac{D}{2}$$

so that

$$V_G = \frac{2}{\pi} = \frac{GsI_e}{(D)(D+2G)} \text{ volts}$$

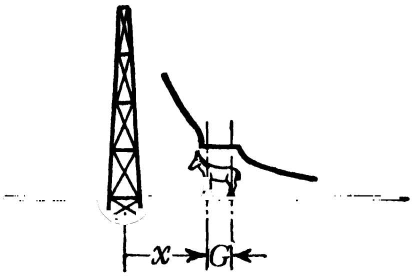


Fig. 4

As regards danger to life from such potential differences, experiment has shown that it is the *current* through the animal which is the criterion of its danger. The magnitude of the short-circuit current so obtained is dealt with in Example 4.

4.—A man is walking in the neighbourhood of a live transmission mast. Find the magnitude of the current which will pass through his body.

ANSWER.—Suppose the foot of the man is taken to be equivalent to a thin flat disc of diameter *d* cms. and as an extreme case assume that the resistance of the man's body can be taken to be relatively of neglibly small value, then from the results given in the Table (see solution to Question 1), the diffusion resistance of such a disc will be

$$R_u = \frac{1}{C_D} = \frac{2\pi}{\varepsilon d} = \frac{s}{2d}$$

where C_D is the capacitance of the electrostatic field due *to one side of the thin disc* (see Chapter IV, page 103, Example 5, Fig. 6), and $s = \frac{4\pi}{\varepsilon}$. Then for the man's two feet in series the total diffusion resistance will be $2R_u$ so that the p.d. V_G which is short-circuited by the man's body will be

$$V_G = 2R_uI_G = \frac{s}{d} I_G$$

But it has been shown in Example 3 that

$$V_G = \frac{G}{x(x+G)} \times \frac{sI_e}{2\pi}$$

where G cm. is the length of the man's stride so that, combining these two equations,

$$I_G = \frac{Gd}{x(x+G)} \times \frac{I_e}{2\pi}$$

If the distance x from the live mast is great in comparison with d then

$$\frac{I_G}{I_e} = \frac{1}{2\pi} \times \frac{Gd}{x^2}$$

If, however, one of the man's feet rests at the extremity of the equivalent diameter D so that $x = \frac{D}{2}$ then the short-circuit current will have its maximum value, viz.,

$$\frac{(I_G)_{\text{max.}}}{I_e} = \frac{2}{\pi} \frac{Gd}{D(D+2G)}$$

If the man stands with one foot at the extremity of the diameter D and touches the mast with his hand, then the diffusion resistance of the earth becomes R_u instead of $2R_u$ as in the foregoing cases, and the short-circuit current will then have twice the value given by the foregoing equation.

Suppose $D = 2$ metres and the stride $G = 1$ metre, and take $d = 0{\cdot}2$ metres, then

$$\frac{(I_G)_{\text{max.}}}{I_e} = \frac{2}{\pi} \frac{Gd}{D(D+2G)} = \frac{2}{\pi} \left\{ \frac{1 \times 0{\cdot}2}{2(2+2)} \right\} = 1{\cdot}6\%$$

If 0·1 ampere is taken to be a fatal current, then the earth current I_e flowing from the live mast must be

$$I_e < \frac{0{\cdot}1}{0{\cdot}016} < \frac{1}{0{\cdot}16} < 6{\cdot}3 \text{ amperes}$$

to be safe.

The danger-zone X in the neighbourhood of the mast may be obtained from the equation

$$\frac{I_G}{I_e} = \frac{1}{2\pi} \frac{Gd}{X^2} \text{ so that } X = \sqrt{\frac{Gd}{2\pi} \frac{I_e}{I_G}}$$

or, taking the foregoing numerical values and a mast earth current of 100 amperes, then

$$X = \sqrt{\frac{1 \times 0{\cdot}2 \times 100}{2\pi \times 0{\cdot}1}} = 5{\cdot}7 \text{ metres.}$$

For large transmission systems the earth current may be as high as 1,000 amperes and the fatal range for such a current will be a radius of about 18 metres round the mast; a fatal shock may then be obtained anywhere within that region. If the earthing of the mast is made deeper so that the centre of the equivalent hemispherical electrode is correspondingly deeper, the range of the danger zone may be somewhat narrowed.

5.—A sphere of 1 cm. diameter is charged with 5 electrostatic units of positive electricity and is suspended by an insulating support at a height of 1 metre above a large flat metal plate placed on the ground. Find the force exerted on the metal plate and the potential of the sphere.

Answer.—Referring to Fig. 5, which shows two spheres of equal radius and with equal and opposite charges of quantity Q electrostatic units. The force on one electrostatic unit of electricity at P distant x cm. from the centre of the upper sphere and $(a - x)$ cm. from the centre of the lower sphere, is

$$F = \frac{Q}{x^2} + \frac{Q}{(a-x)^2} \text{ dynes}$$

The work done in moving this unit charge from the surface of the upper

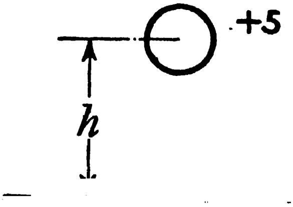


Fig. 6.

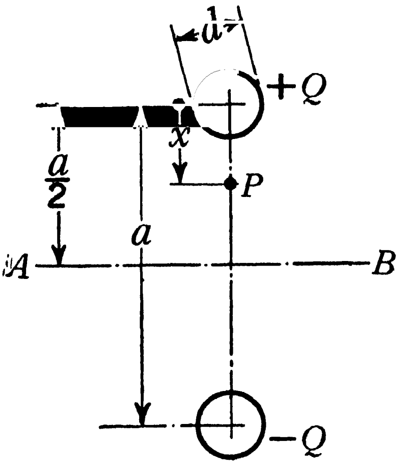


Fig. 5.

sphere to the surface of the lower sphere, that is from $x = d/2$ to $x = (a - d/2)$, is

$$V = \int_{d/2}^{a\text{-}d\ 2} \left(F\,dx \right) = Q\,\frac{8a}{d(2a-d)} \text{ ergs}$$

If d is small with respect to a, this expression becomes

$$V = 4\,\frac{Q}{d} \text{ ergs,}$$

and this is the p.d. between the two spheres expressed in electrostatic units.

In Fig. 6 is shown a metal sphere of diameter d cms., fixed with its centre h cm. above a large flat metal plate, so that the electric field in Fig. 6 will be precisely the same as the electric field in Fig. 5 between the upper sphere and the plane of symmetry $A\ B$ where h in Fig. 6 equals $\frac{a}{2}$ in Fig. 5. It follows, therefore, that the force between the sphere and metal plate of Fig. 6 will be the same as the force between the two spheres of Fig. 5, that is,

$$F = \frac{Q^2}{a^2} = \frac{5^2}{(2h)^2} = \frac{25}{4 \times 10^4} = 6{\cdot}25 \times 10^{-4} \text{ dynes.}$$

The potential of the sphere in Fig. 6 will be one-half the p.d. between the two spheres of Fig. 5, i.e.,

$$V = \tfrac{1}{2}\left(4\frac{Q}{d}\right) = \tfrac{1}{2}\left(\frac{20}{1}\right) = 10 \text{ elec.-st. units} = 3{,}000 \text{ volts.}$$

These results should be compared with those given in the solution of Example 6, Chapter IV, page 47.

6.—Two concentrated quantities each of 2 electrostatic units of positive electricity are placed at 5 cms. apart. Plot the traces of the equipotential surfaces and the lines of force.

Answer.—In Fig. 7 the potential at P due to the positive charges at A and B respectively is

$$\frac{Q}{\varepsilon r} + \frac{Q_1}{\varepsilon r_1}$$

so that the equipotential surfaces will be given by the equation

$$\frac{Q}{r} + \frac{Q_1}{r_1} = \text{a constant.}$$

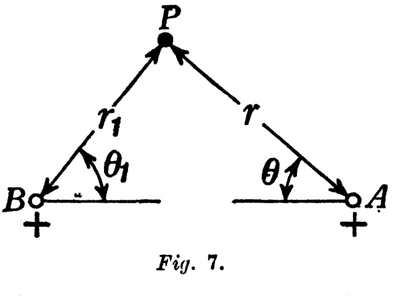


Fig. 7.

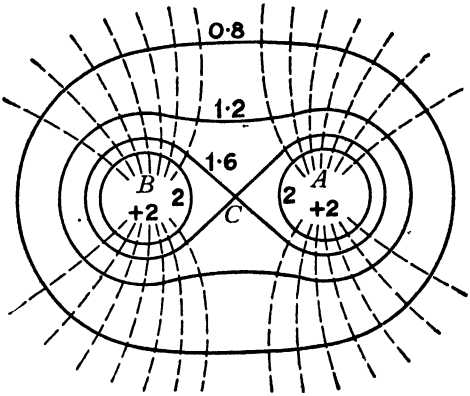


Fig. 8.

If the two charges are equal, this equation becomes

$$\frac{1}{r} + \frac{1}{r_1} = \text{a constant}$$

In Fig. 8 the traces of the equipotential surfaces are shown in full lines for the case in which two concentrated electrostatic units of positive electricity are placed at A and B respectively, where $AB = 5$ cms.

The lines of force are given by the equation (see Fig. 7)

$$\cos\theta - \cos\theta_1 = \text{a constant}$$

and are shown by broken lines in Fig. 8. The derivation of this equation for the lines of force will be clear from page 81, "Principles of Electrical Engineering." The point C is midway between the points A and B and the equipotential surface through this point forms a double symmetrical loop, the potential of points on this loop being 1·6 electrostatic units. For points at a higher potential than 1·6 units the equipotential surfaces are two separate surfaces, viz., one in each part of the double loop through C.

7.—Give an account of the main types of insulation materials for heavy and light current work respectively. Describe the principle of one method of testing the insulation strength by means of high tension rectified alternating current pressure.

ANSWER.—For the reliable and economical operation of electrical machines and apparatus, a fundamental requirement is the choice of suitable dimensions and the economic utilisation of the insulation material. From the point of view of electric strength, the quality of the insulation is ascertained by means of electric breakdown tests.

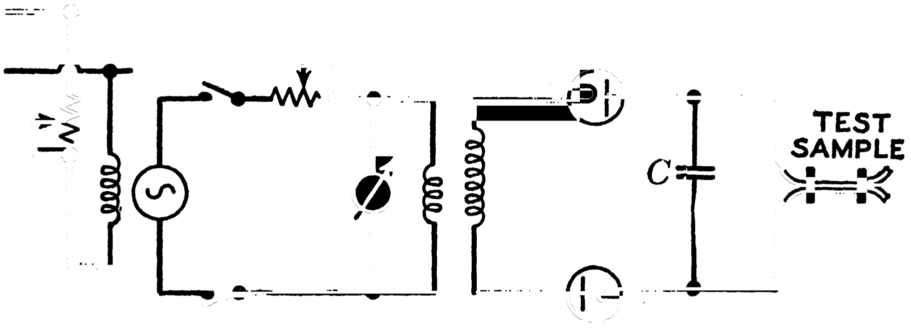


Fig. 9.

The most commonly used materials for heavy current work for the insulation of conductors is cellulose in the form of paper or cotton. In addition, for small motors, and the tiny motors which are built for special services, enamelled wire is available, whilst for high temperature requirements, asbestos, mica, and heat resisting enamel can be used. For light current work, according to the purpose in view, the insulation materials available are paper, cotton, artificial silk, enamel, rubber, and, latterly, synthetic sheet materials such as zellophone.

The insulation material is applied to the conductor in the form of thread, tape, or sheet, usually in several layers bound together with suitable adhesive material. The insulation material may also be applied by fusion or by pressure. For the requirements of special conditions of operation, the insulation may be built up from several different materials. The thickness of insulation depends upon the material used and on the diameter of the wire, and may be from 0·01 to 1 mm. During the insulating process or subsequently, when the insulated conductor has been assembled in position in the machine or apparatus, the mechanical and electric strengths of the overall insulation may be increased by embedding in compound or in insulating oil.

In order to test the electric quality of an insulated conductor, samples are subjected to electric breakdown tests. In the case of all solid insulating materials the breakdown strength of the insulated conductor depends upon (i) the type of applied pressure, (ii) the rate and the time for which the pressure is applied, (iii) the type of electrodes used, (iv) the impregnation process which has been used, (v) the temperature of the conductor, (vi) the thickness of the insulation.

In Fig. 9 are shown the circuit connections for insulation testing by means of rectified alternating current pressure. The secondary winding of the alternating current transformer is connected through two rectifying thermionic valves (diodes), and a smoothing condenser is shown at *C*. The breakdown strength can be measured by means of the voltmeter on the low tension side of the transformer, provided the whole equipment has been previously calibrated. Otherwise, the breakdown pressure may be measured at the rectified A.C. terminals by means of a spherical or plate spark-gap equipment.

8.—Give an account of the electron theory of the action of a crystal detector.

ANSWER.—The crystal detector for the reception of electromagnetic waves was discovered by Dunwoody in 1906 and, on account of its simplicity, sensitivity and cheapness has been very widely used by amateurs for radio reception purposes. It was found that carborundum (a carbide of silicon) crystals had the property that when in contact with another conductor the conductivity across the contact was greater in one direction than in the other and, consequently, it could be used as a rectifier. Subsequently, it was found that many other pairs of conductors had the same property.

The explanation of this effect is not yet fully understood, but the most satisfactory theory yet advanced is that of Schottky which can be briefly explained by reference to a mechanical model as follows. In Fig. 10 is shown a shallow dish in which are a number of small shot in a state of continual movement. If some of these tiny balls have a higher velocity than others, only those which have a sufficient velocity will be able to reach the rim and to escape. In order to enable a greater number to escape it is necessary either to increase their velocity or to reduce the depth of the dish.

The dish in this model represents the inequalities in the surface of a conductor, and the tiny shot represent the electrons. If an electric field is impressed on the conductor, the velocity of the electrons will be increased so that more will be able to reach the rims of the hollow places. For a completely smooth surface, for example, in the case of tungsten, a field strength of $1{\cdot}4 \times 10^8$ volts per cm. is necessary for this purpose. Actually, however, such a completely smooth surface is not realizable in practice, and hollows of 10^{-5} cm. can be detected by the microscope, whilst it is certain that there is a profusion of such depressions of depth less than 10^{-6} cm. Owing to the concentration of the electric field at the edges of a conductor, a smaller average applied electric field intensity than would otherwise be necessary is sufficient to produce an overflow of electrons, viz., for tungsten, an intensity of about $1{\cdot}8 \times 10^7$ volts per cm.

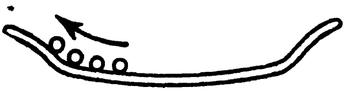


Fig. 10.

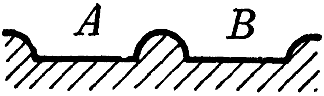


Fig. 11.

In a detector, two electrodes are in extremely close contact and, according to Schottky, the action may be represented by the use of two shallow dishes, which are so close together that their rims coalesce as shown in Fig. 11. If the depth of the two dishes are the same, then an equal number of electrons will pass from A to B as will pass in the opposite direction. If, however, one dish is deeper than the other (say, A is deeper than B), then more shot will pass from B to A than in the reverse direction.

In the contact of two dissimilar conductors, the free energy of the electrons corresponds to the depth of the dish, and accordingly more electrons will be able to pass in one direction than in the other. By altering the strength and direction of the impressed electric field, the effect corresponds to an alteration in the relative depths of the two adjacent dishes of Fig. 11. This electron theory satisfactorily accounts for a large number of observed facts of the operation of such detectors, but requires a modification in some respects to be completely in accordance with the observed phenomena.

Chapter IV

CAPACITANCE : DIELECTRIC CONSTANT: ENERGY OF THE ELECTRIC FIELD.

1.—A condenser discharges through a resistance and spark-gap in series. Obtain the current and pressure relationship showing how the time taken for the condenser to discharge may be found.

ANSWER.—When the pressure of the condenser *C* of Fig. 1 is equal to the ignition pressure of the spark gap (see Example 16, Chapter II), the arc will start across the gap and the subsequent course of events will be as seen from the following considerations. In Fig. 2 is shown the arc characteristic with its two branches *A B* and *B C*, where *A* defines the ignition pressure v_{ig} and *C* the quenching pressure v_{qu}. The resistance characteristic, viz., $v_R = iR$ is shown by *O D* and is drawn below the abscissa for convenience. The pressure across the condenser at every moment will be defined by the equation $v_C = v_R + v_{arc}$. At the instant of ignition the arc pressure will immediately fall (at a very high speed) from the ignition pressure to the value defined by the point *B*, so that the broken-line curve *A B* of Fig. 2 will be traversed in a time which may vary from a few millionths of a second to a few thousandths of a second and the arc proper will start at the point *B* with practically the full ignition pressure, the current then having its maximum value *I* amperes. The condenser pressure will then fall due to the flow of current, and the full-line branch of the characteristic will be traversed up to the point *k* at which point the condenser pressure reaches its minimum value, that is to say the intercept v_{co} between the arc characteristic and the resistance line *O D* reaches a minimum. The arc current then also reaches its

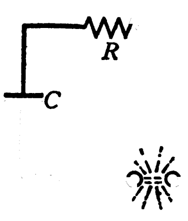


Fig. 1

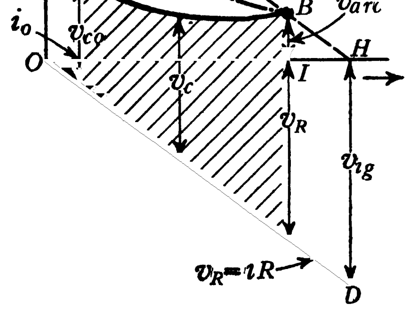


Fig. 2

minimum value since any further fall of current would mean a rise of the condenser pressure which is impossible, and consequently the arc goes out. There will then be a residual charge left in the condenser corresponding to the pressure v_{co}.

The time taken for the whole sequence of events to be accomplished can be determined graphically as shown in the diagrams of Fig. 3. Thus, the quantity of electricity in the condenser at any moment t will be $q = Cv_c$ so that the current will be

$$i = \frac{dq}{dt} = C\frac{dv_c}{dt} \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

and consequently, the time for which the arc persists is

$$t = C\int_{v_{ig}}^{v_{co}} \frac{dv_c}{dt} \quad \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

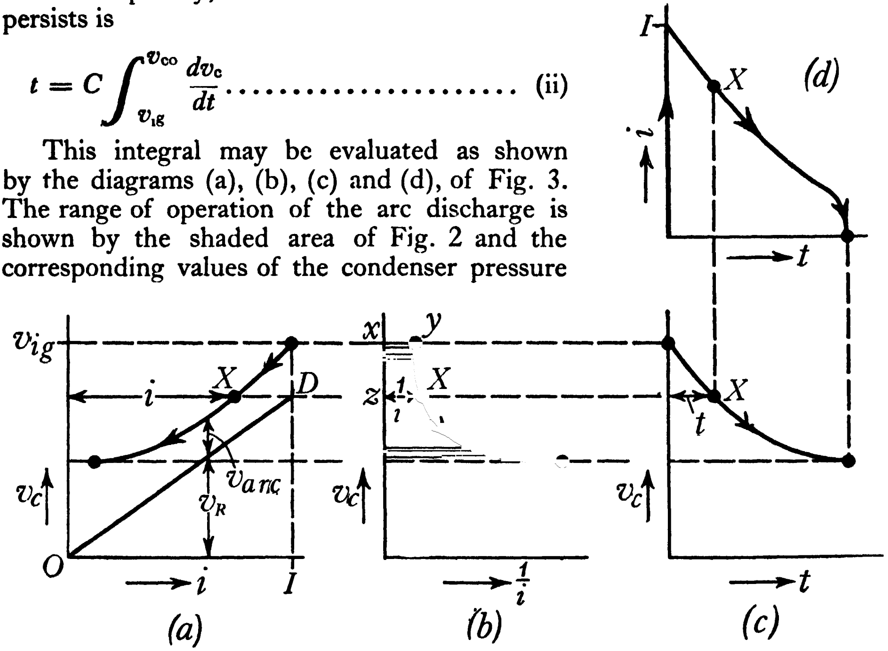


Fig. 3

This integral may be evaluated as shown by the diagrams (a), (b), (c) and (d), of Fig. 3. The range of operation of the arc discharge is shown by the shaded area of Fig. 2 and the corresponding values of the condenser pressure v_c are shown in Fig. 3 (a) as a function of the current: the resistance characteristic line being shown by $O\,D$. In Fig. 3(b) the condenser pressure v_c is given as a function of $\frac{1}{i}$ so that, for any point X, the corresponding time occupied is given by the shaded area $x\ y\ X\ z$ and the value of this area is then $\int_0^t \frac{dv_c}{i}$ and is consequently proportional to the time as defined by the equation (ii). In Fig. 3(c) the time so found is plotted as a function of *the condenser pressure and the current/time curve of Fig. 3(d) is then* easily derived from the graphs of Figs. 3(a) and 3(c). For all the four diagrams the point X for example is a corresponding point.

2.—A condenser is charged from a source of direct current by means of an arc across the switch gap. Give an account of the current and pressure relationships and derive an expression for the time taken to charge the condenser.

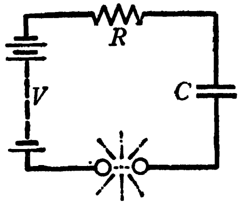


Fig. 4

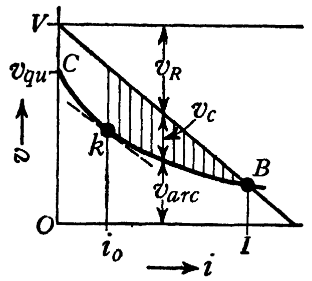


Fig. 5

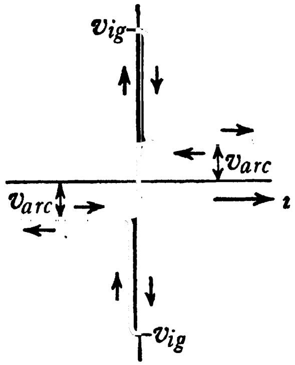


Fig. 6

Answer.—This type of condenser charging process (Fig. 4) is met with in practice in high tension circuits when a spark jumps across the contacts as the switch is being closed. As in Example 1, the arc characteristic is shown by the curve $B\,C$ in Fig. 5 and the resistance characteristic line $O\,D$ of Fig. 1 is shown by $V\,B$. The relationship between the applied pressure V and the component pressures of the circuit is given by the equation,

$$V = v_R + v_c + v_{arc}$$

so that,

$$v_c = V - v_R - v_{arc} = V - Ri - v_{arc} \quad \text{..............(i)}$$

The arc will go out when the condenser pressure v_c becomes a maximum and that value will be reached at the point k at which the tangent drawn to the characteristic curve is parallel to the resistance line characteristic $O\,D$. If the arc were to persist beyond this point that would imply a decrease of condenser pressure and a consequent reversal of the direction of the current, which is impossible. When the arc is extinguished at the point k it is seen from the diagram that the condenser pressure is very much less than the applied pressure. The time required for the charging process corresponds to the shaded area in Fig. 5 and this can be evaluated by graphical integration as in Example 1, Fig. 3.

When the applied pressure is alternating, the arc characteristic may be taken to be of the form shown in Fig. 6, in which case the arc pressure is constant except just at the moment of ignition. In this case the time relationships can be calculated, thus:

$$v_c = \frac{1}{C}\int i\,dt : \frac{1}{C}\,i = \frac{dv_c}{dt} = -R\frac{di}{dt} \quad \text{.......................} \quad \text{(ii)}$$

since V and v_{arc} are now both constant. Hence, from (ii)

$$dt = -RC\frac{di}{i}$$

$$\text{and } t = -RC\,.\,\log_e i\Big]_I^i = -RC\,.\,\log_e\frac{i}{I}$$

$$\text{so that } \frac{i}{I} = e^{-t/\tau} : \text{ where } \tau = RC \text{ (the time constant)}$$

$$i = Ie^{-t/\tau} \text{ where } I = \frac{V - v_{arc}}{R} \quad \text{..............................} \text{(iii)}$$

The condenser pressure will then be given by the expression

$$v_c = V - v_{arc} - RIe^{-t/\tau} = (V - v_{arc})(1 - e^{-t/\tau})$$

When $\frac{t}{\tau} = \pi : e^{-t/\tau} = 0{\cdot}045$, so that the condenser is then charged to within 4·5 % of its final value and this time will therefore about correspond to the condition defined by the point k in Fig. 5. That is to say, the charging time will be of extremely short duration. See also Chapter IX, Fig. 44.

3.—*Derive an expression for the pressure distribution along a string of high tension insulators. (NOTE.—To answer this question knowledge of the contents of Chapter XV is desirable.)*

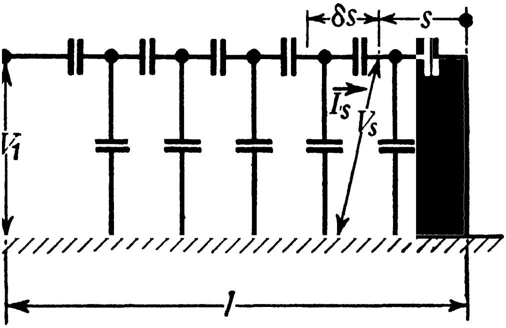


Fig. 7

ANSWER. — A diagrammatical representation of a string of high tension insulators is given in Fig. 7. The capacitance to earth for the total length of the string is C_g farad, so that for the element of length δs the capacitance will be $C_g\frac{\delta s}{l}$ farad. The capacitance of the total series connection of condensers is C_w farad, so that, for an element of length δs the capacitance will be $C_w\frac{1}{\delta s}$. The increase of the pressure vector $\delta\mathfrak{B}_s$ in the elementary increase of length δs will be

$$\delta\,\mathfrak{B}_s = -\,j\,\frac{\delta s}{\omega C_w l}\,\mathfrak{J}_s, \text{ so that } \frac{d\mathfrak{B}_s}{ds} = -j\,\frac{1}{\omega C_w l}\,\mathfrak{J}_s \quad \text{.....................} \quad \text{(i)}$$

where ω is the circular frequency of the sinusoidally varying pressure $\mathfrak{V}_l$ *which is applied to the system at the end of the string, that is the phase* pressure.

Similarly, the increase of current $\delta\mathfrak{J}_s$ in the elementary increase of distance δs will be

$$\delta\mathfrak{J}_s = j\omega C_g \frac{\delta s}{l}\mathfrak{V}_s, \text{ so that } \frac{d\mathfrak{J}_s}{ds} = j\omega\frac{C_g}{l}\mathfrak{V}_s \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

Differentiating (i) with respect to s and substituting from (ii) gives

$$\frac{d^2\mathfrak{V}_s}{ds^2} = \frac{C_g}{C_w l^2}\mathfrak{V}_s \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

The general solution of this equation is

$$\mathfrak{V}_s = \mathfrak{A}e^{\alpha\frac{s}{l}} + \mathfrak{B}e^{-\alpha\frac{s}{l}} \quad \ldots\ldots\ldots\ldots \text{(iv)}$$

where the vectors $\mathfrak{A}$ and $\mathfrak{B}$ are arbitrary constants depending upon the terminal conditions. Differentiating (iv) twice with respect to s and substituting from (iii) gives

$$\frac{d^2}{l^2} = \frac{C_g}{C_w l^2}, \text{ so that } \alpha = \sqrt{\frac{C_g}{C_w}}$$

Since the end of the string is earthed, it follows that $\mathfrak{V}_s = 0$ for $s = 0$ and $\mathfrak{V}_s = \mathfrak{V}_1$ for $s = l$, from which it is easily seen that $\mathfrak{A} = -\mathfrak{B}$. Also by substitution in (iv) it is seen that

$$\mathfrak{A} = -\mathfrak{B} = \frac{\mathfrak{V}_1}{e\alpha - e^{-\alpha}}$$

Then from (iv),

$$\mathfrak{V}_s = \mathfrak{V}_1\left(\frac{e^{\alpha\frac{s}{l}} - e^{-\alpha\frac{s}{l}}}{e^{\alpha} - e^{-\alpha}}\right) = \mathfrak{V}1\,\frac{\text{Sinh}\,\alpha\frac{s}{l}}{\text{Sinh}\,\alpha} \quad \ldots\ldots\ldots\ldots \text{(v)}$$

as is shown by the curve 1 of Fig. 8. The potential gradient is then,

$$\frac{d\mathfrak{V}_s}{ds} = \mathfrak{V}_1\frac{\alpha}{l}\,\frac{\text{Cosh}\,\alpha\frac{s}{l}}{\text{Sinh}\,\alpha}$$

and the maximum gradient will occur when $\frac{s}{l} = 1$, viz.,

$$\left.\frac{d\mathfrak{V}_s}{ds}\right]_{\max} = \mathfrak{V}_1\frac{\alpha}{l}\,\text{Coth}\,\alpha$$

It is also of interest to note that, for small values of α and, in the limit, for $\alpha = 0$: Cosh $\alpha\frac{s}{l} \simeq 1$: Sinh $\alpha \simeq \alpha$. The pressure gradient is then

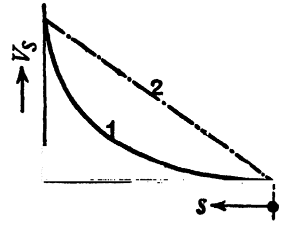


Fig. 8

$\frac{d\mathfrak{V}_s}{ds} = \mathfrak{V}_1 \frac{1}{l}$ and is constant throughout the line as is shown by the chain-dotted line 2 in Fig. 8. (See also: "Engineering," May 22, 1942, page 401.)

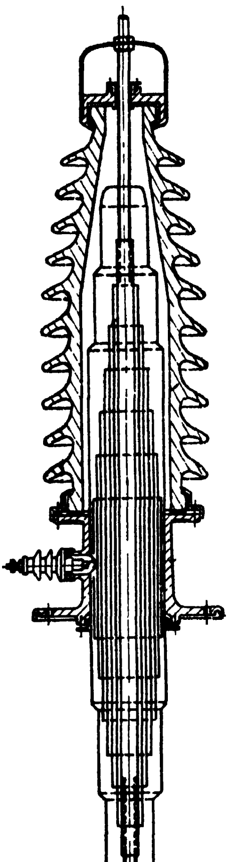


Fig. 9

An important practical example of a condenser chain for controlling the pressure gradient is the condenser type of bushing as used for high tension transformers. With such a constructional type it is possible to tap off a definite percentage of the total pressure for measurement purposes. Fig. 9 shows an example of a condenser type of bushing with a tapping point.

The condenser bushing is constructed of layers of paper treated with synthetic resin varnish and having sandwiched layers of aluminium foil. Such condenser bushings are used on the British Grid for the 132 kV. transformers, and have several advantages over the oil-filled porcelain shell type. Although they are more expensive, condenser type bushings are smaller and lighter, and the reason for this is that the axial pressure gradient at the ends of the bushing can be controlled.

It will be seen from Fig. 9 that the bands of aluminium foil are progressively of shorter axial length as the distance from the axis of the conductor increases. This gives a graded capacitance chain such that the capacitance is relatively large near the surface of the conductor and diminishes towards the outer periphery of the layers of paper insulation. The result is that the pressure gradient is reduced in the neighbourhood of the conductor surface and is increased in the neighbourhood of the outer layers of the insulation, this approximating to the straight line relationship of Fig. 8.

4.—Express the relationship

Q in coulombs = (V in volts) × (C in farads)

in electromagnetic units.

ANSWER.—(V in volts) × (C in farads) = (Q in coulombs).

$$\text{or, } \left(\frac{V \text{ in electromagnetic units}}{10^8}\right) \times \left(\frac{C \text{ in electrostatic units}}{9 \times 10^{11}}\right) =$$

$$10\ (Q \text{ in electromagnetic units})$$

$$\text{so that, } V \text{ in em.units} = Q \text{ in em. units} \left(\frac{1}{C \text{ in es. units}}\right) 9 \times 10^{20}$$

hence, V *in em. units* $= Q$ *in em. units* $\left(\dfrac{1}{C \text{ in em. units}}\right)$

where C in em. units $=$ (C in es. units) $\times 9 \times 10^{20} = c^2 \times$ (C in es. units) and $c = 3 \times 10^{10}$ cm. per sec., and is the velocity of light in open space.

5.—Find the potential at any point due to a long straight wire which is free in space and is charged with q coulombs per centimetre length. Apply the result to the determination of the capacitance per kilometre of a symmetrically arranged three-phase transmission line.

ANSWER.—If a long straight wire free in space is charged with an electric quantity of q coulombs, i.e., $q\,(3 \times 10^9)$ electrostatic units, per cm. length, the electric force at a distance R cms. from the axis of the wire will be :

$$F = \frac{2q}{R} \times 3 \times 10^9 \text{ dynes} = \frac{2q}{R}\,(9 \times 10^{11}) \text{ volts/cm.}$$

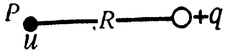


Fig. 10

since 1 electrostatic unit of pressure = 300 volts. Now the electric force in any direction is measured by the rate of fall of potential in that direction so that if u is the potential at the point P Fig. 10, then,

$$-\frac{du}{dR} = F = \frac{2q}{R}\,(9 \times 10^{11}) \text{ volts/cm.}$$

so that,

$$u = -\,2q \times 9 \times 10^{11} \log_e \frac{R}{A} \text{ volts} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

where A is an integration constant which will be defined later.

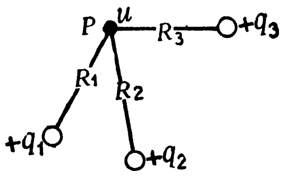


Fig. 11

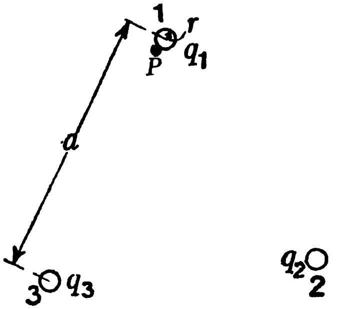


Fig. 12

If there are several charged wires 1, 2, 3 . . ., as shown in Fig. 11, the potential u at any point P is obtained by super-posing the potentials at P due to each of the charged wires, viz.,

$$u = -\,[2q_1 \log_e R_1 + 2q_2 \log_e R_2 + 2q_3 \log_e R_3]\; 9 \times 10^{11} + A^1 \ldots\text{(ii)}$$

where A^1 is an integration constant. As a practical example of the application of these results, consider three symmetrically arranged wires as shown in

Fig. 12 connected to a three-phase supply. The potential at the surface of conductor 2

$$u_p = -2\left[q_1 \log_e \frac{a}{B} + q_3 \log_e \frac{a}{B} + q_2 \log_e \frac{r}{B}\right] 9 \times 10^{11} \text{ volts}$$

$$= -2 \times 9 \times 10^{11}\left[q_1 \log_e \frac{a}{B} + q_2 \log_e \frac{a}{B} + q_3 \log_e \frac{a}{B} + q_2 \log_e \frac{r}{B} - q_2 \log_e \frac{a}{B}\right]$$

and since, for a symmetrical three-phase supply pressure $q_1 + q_2 + q_3 = 0$, this expression reduces to $u_p = 2 \times 9 \times 10^{11} \times q_2 \log_e \frac{a}{r}$ volts : so that the capacitance per phase is

$$C = \frac{q_2}{u_p} = \frac{1}{\left(2 \log_e \frac{a}{r}\right) \times 9 \times 10^6} \text{ farad per km.} \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

since q_2 is the charge per cm. This formula for C will be seen to be the same as that given on page 105 of "Principles of Electrical Engineering."

6.— Two spheres each of radius r cm. and distant a cm. apart in air, are equally and oppositely charged with Q electrostatic units. Find :

(i.) The potential difference between the spheres.

(ii.) The capacitance of the condenser formed by the two spheres.

(iii.) The energy stored in this charged condenser.

(iv.) The electric force acting on each sphere.

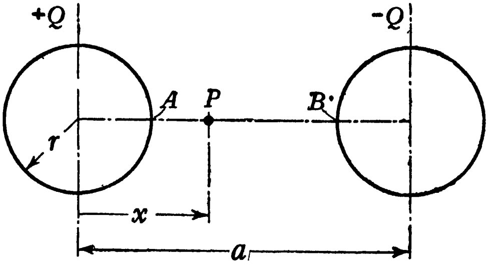


Fig. 13

ANSWER.—If the distance apart a is large as compared with the radius r of each sphere, it may be assumed that the action is the same as if the charges Q were concentrated at the centres of the respective spheres.

The force on a unit positive charge at a point P (Fig. 13) will be :

$$\frac{Q}{x^2} + \frac{Q}{(a-x)^2} \text{ dynes}$$

and the work done by the electric field when the unit charge is moved from A to B is

$$V = Q \int_{x=r}^{x=(a-r)} \left[\frac{1}{x^2} + \frac{1}{(a-x)^2}\right] dx = Q\left[-\frac{1}{x} + \frac{1}{a-x}\right]_r^{a-r}$$

that is.

$$V = 2Q \frac{a-2r}{r(a-r)} \text{ e.s. units} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

The capacitance of the condenser is then

$$C=\frac{Q}{V}=\frac{r\,(a-r)}{2\,(a-2r)}\ e.s.\ \text{units} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

The energy stored in this charged condenser is

$$U=\tfrac{1}{2}\,Q.V=\tfrac{1}{2}\,\frac{Q^2}{C}=Q^2\,\frac{(a-2r)}{r\,(a-r)}\ \text{ergs} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

The electric force on each sphere is

$$F=-\frac{d}{da}\ U=Q^2\,\frac{1}{r(a-r)}\ \left[\frac{r}{a-r}\right]\ \text{dynes}$$

and, when r is small in comparison with a

$$F \simeq \frac{Q^2}{a^2}\,\text{dynes} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

This solution should be compared with that of Question 5, Chapter III.

7.—Three long straight horizontal and parallel wires are supported at different heights above the earth's surface and each is charged with q coulombs per centimetre length. Find the potential at any point and apply the result to the calculation of the capacitance per phase of a symmetrically arranged three-phase transmission line.

ANSWER.—In example 5 it was assumed that the wires were suspended in free space. Transmission lines, however, cannot be dealt with as though they were far removed from all other conductors, since they are in the neighbourhood of the earth's surface, and the influence of this surface on the capacitance must be taken into account. This is of special importance when an earth wire is carried by the transmission line masts. The effect of the earth conductivity can be taken into account quite easily by means of the principle of electric images. That is to say, if in Fig. 14 a line is assumed to run horizontally at a height h above the ground, the force at any point P due to an electric charge of q coulombs per cm. length of the line will be the same as that due to the combined effects of the line G and its image G' in the earth's surface. In this way the conductivity of the earth's surface is taken into account and the wire G charged with $+q$ coulombs per cm. and its image G' charged with $-q$ coulombs per cm. can then be assumed to be situated in free space. Thus, the potential at P will be

$$u_P=[-2q\log_e R_1+2q\log_e R_2]\ 9\times10^{11}+A^1 \ \ldots\ldots\ldots\ldots\ \text{(i)}$$

For any point Q on the earth's surface the potential will be

$$u_P=[-2q\log_e R'+2q\log_e R']\ 9\times10^{11}+A^1 \ldots\ldots\ldots\ldots\text{(ii)}$$

But the potential at the earth's surface is zero so that the integration constant A^1 in expression (i) and (ii) is also zero. That is to say, the potential at any point P as given by the expression (i) may now be written

$$u_P = 9 \times 10^{11} [- 2q \log_e R_1 + 2q \log_e R_2] \text{ volts} \ldots\ldots\ldots\ldots \text{(iii)}$$

Suppose now that there are several parallel horizontal wires as shown in Fig. 15 the radius of each wire being r_1 cm. It will be assumed that the electric charge on each wire will act as though it were concentrated on its axis, and in general this will be true very closely for the conditions which hold in practice for overhead transmission lines. The potential at the surface of the wire 1 will then be (see Fig. 15),

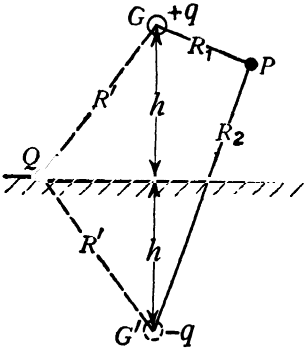


Fig. 14

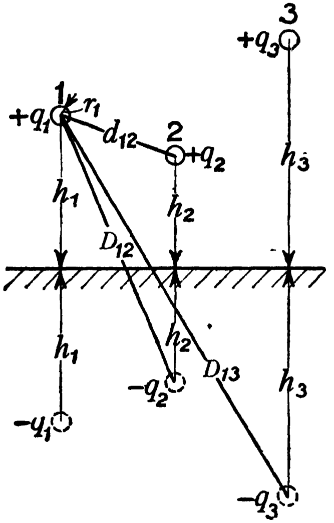


Fig. 15

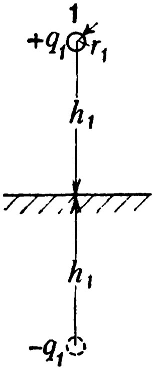


Fig. 16

$$u_1 = 9 \times 10^{11} \left\{ \begin{array}{l} - 2q_1 \log_e r_1 - 2q_2 \log_e d_{12} - 2q_3 \log_e d_{13} \\ + 2q_1 \log_e (2h_1) + 2q_2 \log_e D_{12} + 2q_3 \log_e D_{13} \end{array} \right\}$$

that is

$$u_1 = 9 \times 10^{11} \left\{ 2q_1 \log_e \frac{2h_1}{r_1} + 2q_2 \log_e \frac{D_{12}}{d_{12}} + 2q_3 \log_e \frac{D_{13}}{d_{13}} \right\} \ldots\text{(iv)}$$

Writing (see Fig. 16)

$$a_{11} = 2 \times 9 \times 10^{11} \left[\log_e \frac{2h_1}{r_1} \right]$$

$$a_{22} = 2 \times 9 \times 10^{11} \left[\log_e \frac{2h_2}{r_2} \right]$$

$$a_{33} = 2 \times 9 \times 10^{11} \left[\log_e \frac{2h_3}{r_3} \right]$$

Also (see Fig. 17):

$$a_{12} = 2 \times 9 \times 10^{11} \left[\log_e \frac{D_{12}}{d_{12}}\right] = a_{21} : a_{13} = 2 \left[\log_e \frac{D_{13}}{d_{13}}\right] 9 \times 10^{11} = a_{31}$$

$$a_{23} = 2 \times 9 \times 10^{11} \left[\log_e \frac{D_{23}}{d_{23}}\right] = a_{32}$$

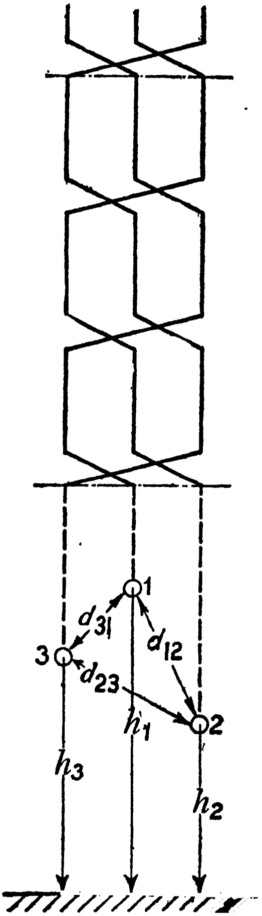


Fig. 19

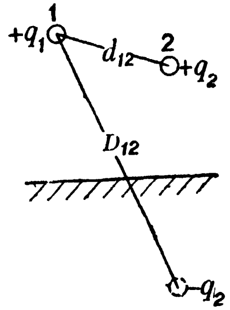


Fig. 17

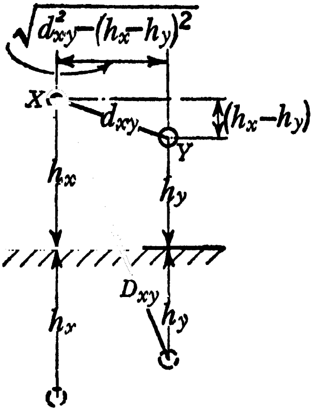


Fig. 18

Then,

$$\left.\begin{array}{l} u_1 = a_{11}q_1 + a_{12}q_2 + a_{13}q_3 \\ u_2 = a_{21}q_1 + a_{22}q_2 + a_{23}q_3 \\ u_3 = a_{31}q_1 + a_{32}q_2 + a_{33}q_3 \end{array}\right\} \dots\dots\dots\dots\dots\dots \text{(v)}$$

The factors a in these equations are termed the "potential coefficients."

In general,

$$a_{xy} = 2 \left[\log_e \frac{D_{xy}}{d_{xy}}\right] 9 \times 10^{11}$$

From Fig. 18 it is seen that,

$$D^2_{xy} = (h_x + h_y)^2 + (d_{xy})^2 - (h_x - h_y)^2 = 4h_xh_y + d^2_{xy}$$

hence,

$$\frac{D_{xy}}{d_{xy}} = \sqrt{\frac{4h_xh_y}{(d_{xy})^2} + 1}$$

so that,

$$a_{xy} = 2 \left[\log_e \sqrt{\frac{4h_xh_y}{(d_{xy})^2} + 1}\right] 9 \times 10^{11}$$

that is,

$$a_{xy} = 9 \times 10^{11} \left\{ \log_e \left[\frac{4h_xh_y}{(d_{xy})^2} + 1\right] \right\} \dots\dots\dots\dots \text{(vi)}$$

In Fig. 19 is shown diagrammatically a general example of a symmetrically " transposed " three-phase overhead transmission line. The purpose of such a system of transposition is to protect neighbouring electric circuits (e.g., telephone lines) from electromagnetic and electrostatic interference to which the high tension lines would otherwise give rise.

In this arrangement, the geometric mean height may be used as the height of each conductor, viz.,

$$h = \sqrt[3]{h_1 \times h_2 \times h_3}$$

Since the radius r of each wire is the same, it follows that,

$$a_{11} = a_{22} = a_{33} = 2\left[\log_e \frac{2h}{r}\right] 9 \times 10^{11}$$

The mean distance between any two wires may be taken as,

$$d = \sqrt[3]{d_{12} \times d_{23} \times d_{31}}$$

so that from equation (vi),

$$a_{12} = a_{13} = a_{23} = 9 \times 10^{11} \log_e \left[\left(\frac{2h}{d}\right)^2 + 1\right]$$

and the system of equations (v) may now be written

$$\left.\begin{aligned} u_1 &= a_{11}q_1 + a_{12}q_2 + a_{12}q_3 \\ u_2 &= a_{12}q_1 + a_{11}q_2 + a_{12}q_3 \\ u_3 &= a_{12}q_1 + a_{12}q_2 + a_{11}q_3 \end{aligned}\right\}$$

These equations may be transformed as follows:

$$\left.\begin{aligned} u_1 &= a_{11}q_1 + a_{12}q_2 + a_{12}q_3 \\ u_1 - u_2 &= (a_{11} - a_{12})\, q_1 + (a_{12} - a_{11})\, q_2 + 0 \\ u_1 - u_3 &= (a_{11} - a_{12})\, q_1 + 0 + (a_{12} - a_{11})\, q_3 \end{aligned}\right\}$$

Multiplying the second and third of these equations by $\dfrac{a_{12}}{a_{11} - a_{12}}$ and adding together the three equations so obtained, gives,

$$u_1 + (u_1 - u_2)\frac{a_{12}}{a_{11} - a_{12}} + (u_1 - u_3)\frac{a_{12}}{a_{11} - a_{12}} = (a_{11} + 2a_{12})q_1$$

or,

$$q_1 = \frac{u_1}{a_{11} + 2a_{12}} + \frac{(u_1 - u_2)}{(a_{11} + 2a_{12})}\,\frac{a_{12}}{(a_{11} - a_{12})} + \frac{u_1 - u_3}{(a_{11} + 2a_{12})}\,\frac{a_{12}}{(a_{11} - a_{12})}$$

Writing,

$$K_{11} \text{ for } \left(\frac{1}{a_{11} + 2a_{12}}\right) \quad : \quad K_{12} \text{ for } \frac{a_{12}}{a_{11} - a_{12}} K_{11}$$

gives

$$\boxed{q_1 = u_1 K_{11} + (u_1 - u_2) K_{12} + (u_1 - u_3) K_{12}} \quad \ldots\ldots\ldots\ldots\ldots \text{(vii)}$$

This equation shows that the electric charge q_1 on conductor 1 can be considered as being due to the superposed effects of (see Fig. 20):

(*a*) The phase pressure u_1 acting across the capacitance K_{11}, and,

(*b*) The line pressures $(u_1 - u_2)\,(u_1 - u_3)$ acting across the capacitance K_{12} between the lines.

The equation (vii) may now be written:

$$q_1 = u_1K_{11} + (2u_1 - u_2 - u_3)\ K_{12}, \ldots\ldots\ldots\ldots \text{(viii)}$$

and since for a symmetrical three-phase supply

$$u_1 + u_2 + u_3 = 0.$$

$$\boxed{q_1 = u_1(K_{11} + 3K_{12})} \ldots\ldots \text{(ix)}$$

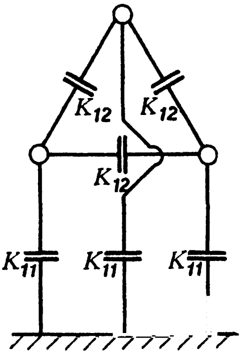


Fig. 20

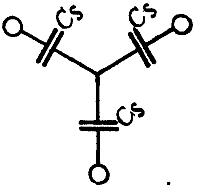


Fig. 21

The capacitance $K_{11} + 3K_{12}$ is the "working capacitance" per phase of the line to the star point: it being observed that the capacitance of a line to the star point is three times the capacitance between any two lines (see Chapter IX, page 305).

Hence (see Figs. 20 and 21),

$$q_1 = u_1{\cdot}C_s$$

$$\text{where } C_s = K_{11} + 3K_{12} = K_{11} + K_{\text{phase}} \ldots\ldots \text{(x)}$$

Taking the following numerical values, viz.,

$$h_1 = 7 \text{ m.} : h_3 = 9\tfrac{1}{2} \text{ m.} : h_2 = 12\tfrac{1}{2} \text{ m.}:$$

$$h = \sqrt[3]{7 \times 9\tfrac{1}{2} \times 12\tfrac{1}{2}} = 9{\cdot}4 \text{ m.}$$

$$d_{12} = 6 : d_{23} = 4 : d_{31} = 4{\cdot}5$$

$$d = \sqrt[3]{6 \times 4 \times 4{\cdot}5} = 4{\cdot}8 \text{ m.}$$

$$r = 0{\cdot}5 \text{ cm.} : \frac{2h}{r} = \frac{1{,}880}{0{\cdot}5} = 3{,}760$$

$$a_{11} = 2\left(\log_e \frac{2h}{r}\right) 9 \times 10^{11} = 2\,(\log_e 3{,}760) 9 \times 10^{11} = 147 \times 10^{11}$$

$$a_{12} = \left\{\log_e \left[\left(\frac{2h}{d}\right)^{\prime} + 1\right]\right\} 9 \times 10^{11} = \left\{\log_e \left[\left(\frac{18{\cdot}4}{4{\cdot}8}\right)^{\prime} + 1\right]\right\} 9 \times 10^{11} =$$

$$25 \times 10^{11}$$

$$\therefore K_{11} = \frac{1}{(a_{11} + 2a_{12})} = \frac{1}{(147 + 50)10^{11}} = \frac{0{\cdot}51}{10^8} \text{ farad per km.}$$

$$3K_{12} = \frac{3 \times a_{12}}{a_{11} - a_{12}} K_{11} = \frac{3 \times 25}{122} \times \frac{0{\cdot}51}{10^8} = \frac{0{\cdot}31}{10^8} \text{ farad per km.}$$

so that the "working capacitance" per phase is

$$\underline{C_s = K_{11} + 3K_{12} = 0{\cdot}82 \times 10^{-8} \text{ farad per km.}}$$

It is of interest to see how the result given by the expression (x) compares with that obtained from the formula which was derived in Example 5, for the condition that the effect of the earth's surface on the capacity was negligibly small. Thus

$$C_s = \frac{1}{9 \times 10^6 \left(2 \log_e \frac{a}{r}\right)} \text{ farad per phase per km.}$$

in which a = 4·8. m. : r = 0·5 cm., so that

$$\frac{a}{r} = \frac{4{,}800}{5} = 960, \text{ hence}$$

$$C_s = \frac{1}{9 \times 10^6 \times 2 \times 2{\cdot}3 \times 2{\cdot}98} = 0{\cdot}816 \times 10^{-8} \text{ farad per phase per km.,}$$

so that the effect of the earth's surface on the capacitance in this case is seen to be very small.

8.—A transformer bushing is built up as follows:—

Layer No.				*Dielectric constant*
0	*Copper rod . . .*	*outside radius*	*2·0 cm.*	
1	*Layer of varnished paper*	*outside radius*	*2·3 cm.*	*4*
2	*Layer of "compound" .*	*outside radius*	*5·0 cm.*	*2.6*
3	*Porcelain shell . . .*	*outside radius*	*7·6 cm.*	*5*

Maximum pressure gradients:—

For the paper insulation . . . . 30,000 volts/cm.
For the "compound" . . . . 40,000 ,,
For the porcelain . . . . . 9,500 ,,

Find the potential of the copper rod and the pressure gradient in the air at the outside surface of the porcelain shell.

ANSWER.—A cross-section of the bushing is shown in Fig. 22. Then for the paper layer (1) of insulation the conditions are:—

$$E_{1max}\, \varepsilon_1\, r_0 = E_{1min}\, \varepsilon_1\, r_1 = E_{1r}\, \varepsilon_1\, r.$$

At the boundary of the paper layer (1) and the compound layer (2) the conditions are:—

$$E_{1min}\, \varepsilon_1\, r_1 = E_{2max}\, \varepsilon_2\, r_1$$

At the boundary of the compound layer (2) and the porcelain layer (3):

$$E_{2min}\, \varepsilon_2\, r_2 = E_{3max}\, \varepsilon_3\, r_2$$

At the boundary of the porcelain shell (3) and the outside air (4):

$$E_{3\min}\,\varepsilon_3\,r_3 = E_{4\max}\,\varepsilon_4\,r_3$$
$$= E_{4\max}\,r_3$$

since for the air $\varepsilon_4 = 1$.

For any radius r within any one layer, e.g., the compound layer (2):

$$E_{2\max}\,\varepsilon_2\,r_1 = (E_2)_r\,\varepsilon_2\,r$$

$$\text{or } E_{2r} = E_{2\max}\frac{r_1}{r}$$

In Fig. 23 is shown the diagram for the pressure gradient across the whole section of the bushing and for the outside air. If the area of this graph which is enclosed between the two ordinates AB and CD is evaluated in (volts per cm. gradient) × cms. unit, the result gives the potential of the copper rod, viz.,
$V_0 = 100{,}000$ volts.

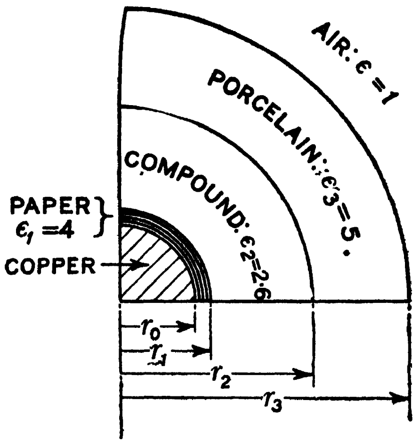


Fig. 22

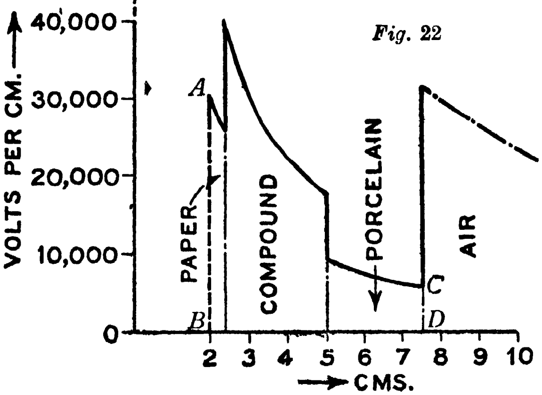


Fig. 23

9.—*What significance has corona phenomena for the design of overhead transmission lines?*

Answer.—From the results given in Chapter III, page 85, "Principles of Electrical Engineering" it is seen that the electric force on a charged element δS of a conductor surface and due to all the rest of the charged surface is

$$F\delta S = \frac{2\pi\sigma^2}{\varepsilon}\,\delta S \text{ dynes.}$$

where σ is the density of the charge in electrostatic units per sq. cm. so that the intensity of the force is

$$F = \frac{2\pi\sigma^2}{\quad} \text{ dynes per sq. cm.}$$

and since the intensity of the electric field is

$$E = \frac{2\pi\sigma}{\varepsilon}$$

it follows that

$$F = \frac{\varepsilon E^2}{8\pi} \text{ dynes per sq. cm.}$$

Taking the breakdown strength of dry air at normal temperature and pressure to be 32,000 volts per cm., that is

$$E = \frac{32{,}000}{300} = 107 \text{ electrostatic units per cm.}$$

so that

$$F = \frac{\varepsilon\, 107^2}{8\pi} = 460 \text{ dynes per sq. cm.}$$

and this quantity has already been defined as the "electrostatic pressure," [see page 85, expression (12)].

If, therefore, the force at the surface of a charged conductor (i.e., the "electrostatic pressure") is greater than about 460 dynes the air ceases to insulate the conductor and the charge escapes.

When the field strength exceeds the breakdown strength of the air an electron stream issues from the conductor (e.g., an overhead transmission line) and ionizes the air in the neighbourhood. This becomes visible as a luminous halo or "corona" surrounding the conductor, and if the pressure further increases, a powerful brush discharge develops which can reach to a distance of a metre or more from the line. Such "corona" effects involve serious energy losses and consequently the diameter of the overhead line must be carefully designed to eliminate the possibility of such brush discharges.

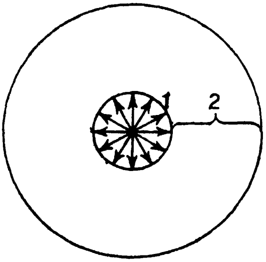


Fig. 24

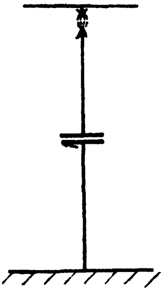


Fig 25.

An immense amount of experimental work has been done in order to obtain complete information regarding the characteristics of corona phenomena on overhead transmission lines, and it is known that the field strength at which the glow discharge appears is not a fixed quantity but depends upon the diameter of the wire, viz., for wires of large diameter the field intensity at which a glow discharge begins may be 20 kV./cm., whilst for wires of small diameter it may be as much as 50 kV./cm.

The effect of corona discharge on an overhead transmission line may be diagrammatically represented as shown in Figs. 24 and 25. The region 1 of Fig. 24 denotes the range of the ionizing effect of the electrons which are discharged from the line and the region 2 marks out the range of the capacitance current and the current due to the travelling ions and electrons. In Fig. 25 the respective ranges are denoted by a spark gap and a capacitance.

It is to be observed, however, that the visible glow discharge is not a criterion of the commencement of energy loss since a dark discharge precedes the glow effect and can give rise to serious energy losses. For specified conditions of smooth wire surface, with dry air, and normal temperature (20° C.) and pressure (760 mm.) the energy loss commences when the field strength is 21 kV./cm. and is independent of the diameter of the wire. In practice, however, these conditions are seldom realized : the wire surface is usually rough, polluted, and weather-worn, the air is not always dry, and rain and mist may envelop the line. The temperature, especially in the summer, may rise well above the normal value of 20° C., whilst the pressure follows the usual barometric variations and at high altitude becomes notably reduced. Fortunately, the unfavourable circumstances seldom appear simultaneously, and it is usual practice to allow a factor of safety of about 30 per cent. in the conductors for overhead transmission lines, provided the altitude is not greater than 500 m. from the sea level datum

10.—What significance has corona phenomena for the design of underground cables ?

Answer.—A dominant factor in the design and reliable operation in service of a high tension cable is the question of corona effects, and one safeguard in this respect is the relatively low values which have been fixed for the permissible temperature rise. Thus, for cables which operate at pressures up to 6 kV. the permissible temperature rise is limited to about 35° C. and for cables operated at higher pressure the permissible rise is about 25° C. The way in which the temperature influences the development of corona discharge may be seen from the following considerations. As the temperature of the cable rises the insulating compound expands at a greater rate than the paper and consequently the compound becomes forced against the lead sheath which yields and stretches to a corresponding extent. When the cable cools down again it draws the compound back with it but cannot reabsorb it completely so that hollow spots may in this way be formed in the body of the paper. Such hollow spaces promote the formation of glow discharges and this in turn, by its mechanical action, drives the oil out of the adjacent paper whilst, at the same time, polymerisation causes the insulation material to disappear and consequently increases the size of the original minute cavities.

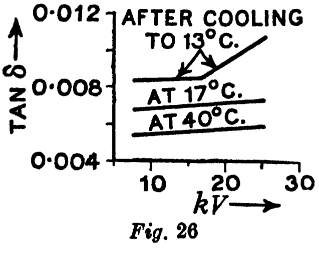


Fig. 26

The permanent stretching of the lead sheath already referred to gives rise to vacant spaces at the inner surface of the sheath and oil gas can form there through which glow discharge can pass. To eliminate this particular danger a method has been evolved for coating the outer layer of the paper insulation with metal foil and in this way prevent any electric field becoming established in such vacant spaces.

The development of glow discharges in cable dielectrics can be experimentally demonstrated by measuring the " loss angle " δ (see Example 12 of this Test Paper and also pages 64 and 362, " Principles." Usually the dielectric loss in a cable is only about 1 per cent. of the copper loss and is quite negligible from the point of view of efficiency tests, the angle of loss provides a very sensitive indicator as to the decay of the dielectric quality. In Fig. 26 is shown the value of tan δ as a function of the applied pressure at temperatures of 17° C. and 40° C. respectively from which it is seen that tan δ changes very little with the applied pressure. When the cable cools down from 40° C. cavities develop in the insulation and if the values of tan δ are then measured as a function of the applied pressure the angle of loss suddenly changes its rate of increase and this sudden change is a sign that a glow discharge has commenced. The characteristic feature of gas- and oil-filled high tension cables is that the construction is such that no hollow spaces can develop in the insulation and consequently, corona effects are eliminated.

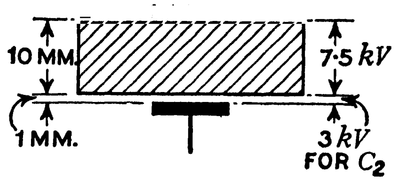
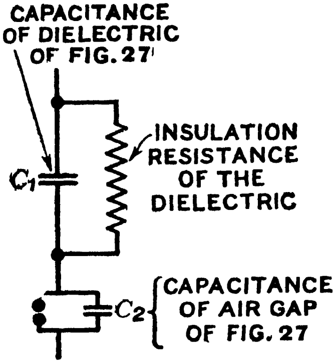


Fig 27 (*above*)

Fig. 28 (*right*)


CAPACITANCE OF DIELECTRIC OF FIG. 27
INSULATION RESISTANCE OF THE DIELECTRIC
C_1
C_2
CAPACITANCE OF AIR GAP OF FIG. 27

A cable structure of core, insulation, and lead sheath can, for experimental purposes, be represented by the arrangement shown in Fig. 27 the equivalent circuit for which is given in Fig. 28. When A.C. is applied to this system glow discharge will develop in the 1 mm. gap when the peak value of the applied pressure wave is greater than about 3 kV., and such a glow discharge corresponds to the discharge of the condenser C_2 in Fig. 28, this condenser becoming charged whilst the pressure is below the critical value. In this way a glow discharge will occur twice during each cycle, that is 100 times per second for a supply frequency of 50 cycles/sec.

11.—*A capacitance is formed from two flat parallel plates as electrodes. A slab of solid insulation material of thickness t_1 mm. and of dielectric constant 4 is close against one electrode and the air-gap between the other electrode and the insulation slab is t_2 mm. wide. If a pressure V is applied across the electrodes, find the pressure drop across the air-gap.*

ANSWER.—If v_1 is the field strength (i.e., the pressure gradient) in the solid insulator and v_2 for the air gap, then at the boundary surface of the two

$$v_1 \varepsilon_1 = v_2 \varepsilon_2 \quad \text{(i)}$$

The pressure drop across the solid insulator will therefore be $v_1 t_1$ and across the air-gap, $v_2 t_2$, so that the total applied p.d. will be

$$V = v_1 t_1 + v_2 t_2 \quad \text{(ii)}$$

Combining (i) and (ii) gives

$$V = v_1\left(t_1 + \frac{\varepsilon_1}{\varepsilon_2} t_2\right) = v_2\left(t_2 + \frac{\varepsilon_2}{\varepsilon_1} t_1\right) \quad \text{(iii)}$$

If $v_2 = 3$kV./mm. $t_1 = 10$ mm. $t_2 = 1$ mm.

$\varepsilon_1 = 4 : \varepsilon_2 = 1$

then $V = 3(1 + \frac{1}{4}\,10) = 7{\cdot}5 + 3 = 10{\cdot}5$ kV.

so that the pressure drop in the air-gap will be 3 kV. and in the solid insulator 7·5 kV. (see also, Example 10, Fig. 28a of this Test Paper).

12.—*A condenser is built up of 11 plates as shown in Fig. 29a, the surface area of each plate is 50 sq. cm. and the distance between adjacent plates is 0.003 cm. If the condenser is charged to a pressure of 250 volts and the dielectric constant is $\varepsilon = 9$, find—*

(i) The capacitance of the condenser.

(ii) The energy stored.

(iii) The force of attraction between two adjacent plates.

ANSWER.—The complete condenser is shown diagrammatically in Fig. 29a, from which it is seen to comprise 5 units in parallel of the dimensions shown in Fig. 29b. The capacitance of this unit is

$$C_1 = \frac{\varepsilon \times 2S}{4\pi d \times 9 \times 10^5}\,\mu\text{F} = \frac{9 \times 100}{113 \times 300} = 0{\cdot}0265\,\mu F$$

so that the complete condenser shown in Fig. 29*a* will have a capacitance

$$\underline{C = 5C_1 = 0{\cdot}1325\,\mu F} \quad \text{(i)}$$

The energy stored in the condenser is

$$U = {}^1/_2 QV = \tfrac{1}{2}V^2C = \tfrac{1}{2}(250)^2 \times 0{\cdot}1325 \times 10^{-6} \text{ joule}$$

that is, $U = 0{\cdot}00415$ joule .... (ii)

The energy stored per ***unit volume*** of dielectric is

$$u = \frac{\varepsilon E^2}{8\pi} \text{ erg}$$

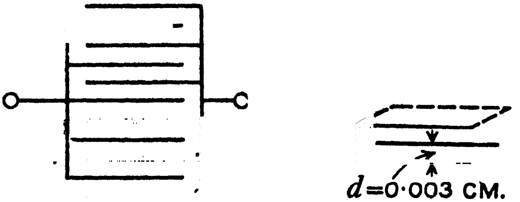
where E is the intensity of the electric field in electrostatic units. That is,

$$E = \frac{\text{Volts per cm.}}{300} \text{ dynes} = \frac{250}{0{\cdot}003 \times 300} = 278 \text{ dynes}$$

so that the total energy stored in the dielectric between two adjacent plates is

$$U = \frac{\varepsilon(278)^2 \times Sd}{8\pi} \text{ ergs}$$

If it be assumed that the distance d between two adjacent plates is increased by the small amount δ, and if the force between these two adjacent plates is P dynes, then



(a) (b)

Fig. 29

$P \times \delta$ ergs will be the work expended when the distance is increased by the amount δ

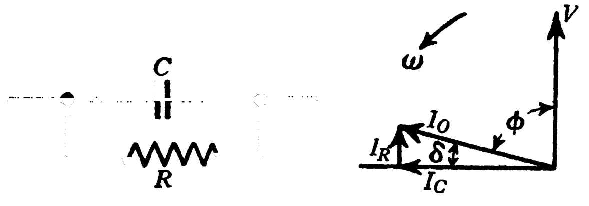
so that
$$P \times \delta = \left[\frac{\varepsilon(278)^2 S(d+\delta)}{8\pi} - \frac{\varepsilon(278)^2 Sd}{8\pi}\right] \text{ergs}$$

from which
$$P = \frac{\varepsilon(278)^2 S}{8\pi} = \frac{9 \times (278)^2 50}{8\pi} = 1{\cdot}38 \times 10^6 \text{ dyne} \;\ldots\ldots\; \text{(iii)}$$

Note.—This solution is identical in form with that for the force of attraction between two magnetised surfaces shown in Examples 7 and 8 of Test Paper VII, pages 101 and 102.

13.—Describe a method for determining the dielectric resistance by measuring the rate at which a charged and insulated cable loses its potential.

ANSWER.—The insulated cable can be represented by a capacitance of C farad in parallel with a resistance of R ohms, as shown in Fig. 30*a*, where C is the capacitance of the cable and R its dielectric resistance. Suppose a single-core cable is connected to an a.c. pressure of V volts at



(a) (b)

Fig. 30

one end, the other end of the cable being insulated. The vector diagram will then be as shown in Fig. 30*b*, where

I_C amperes is the capacitance current
I_R amperes is the resistance current
δ is the angle of loss
$\omega = 2\pi f$ is the circular frequency of the applied pressure.

Then $$I_R = I_C \tan\delta, \text{ so that } \frac{V}{R} = V\omega C \tan\delta$$

or $$R = \frac{1}{\omega\, C \tan\delta} \text{ ohms}$$

If, for example, $C = 5 \times 10^{-6}$ farad : $\tan\delta = 0{\cdot}0063$: $\omega = 2\pi 50 = 314$

then $$R = \frac{10^6}{314 \times 5 \times 0{\cdot}0063} = 10^5 \text{ ohms}$$

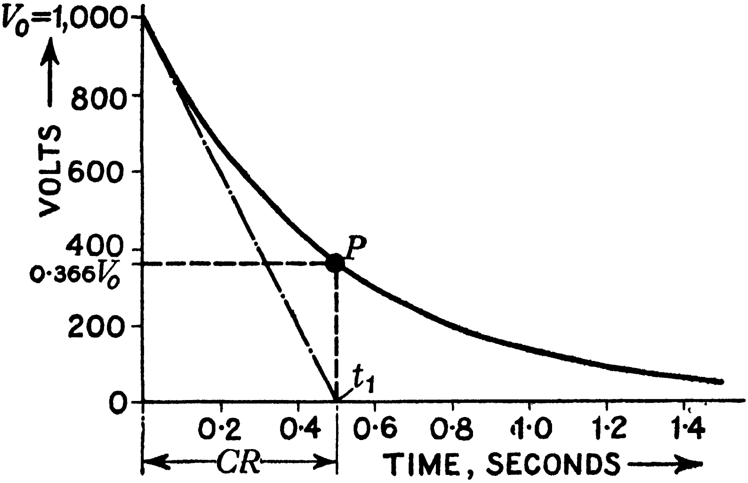


Fig 31

Suppose such a cable is charged by means of a d.c. source to a pressure V_0 volts, an electrostatic voltmeter being connected across the core and sheath and, when the cable becomes charged, the d.c. source is disconnected. The charged condenser will then begin to discharge through the resistance *R*, and at any moment *t* second subsequent to the commencement of the discharge the pressure will be given by the equation

$$v = \left(V_0\right) e^{-\frac{t}{\tau}}$$

where $\tau = CR$ second and is the " electrostatic time constant."

Applying the numerical data given in the foregoing, then

$$\tau = CR = 5 \times 10^{-6} \times 10^{5} = 0{\cdot}5 \text{ sec.}$$

and if $V_0 = 1{,}000$ volts, then

$$v = 1{,}000e^{-2t} \text{ volts}$$

and the pressure/time curve given in Fig. 31 has been drawn to represent this equation.

It is of practical convenience to note that $e = 2{\cdot}718 : \frac{1}{e} = 0{\cdot}366$, so that if t_1 is the time taken for the pressure to fall to $u = 0{\cdot}366V_0 = 366$ volts,

then $$v = 0{\cdot}366V_0 = \left(V_0\right)e^{-\frac{t}{\tau}}$$

or $$0{\cdot}366 = \frac{1}{e} = \frac{1}{e^{\frac{t}{\tau}}}$$

that is, $\tau = t_1 = CR$, and hence

$$R = \frac{t_1}{C} \text{ ohms}$$

so that it is only necessary to obtain t_1 in order to find the resistance when the capacitance is known. The point P in Fig. 31 defines these relationships

14.—*Find the capacitance due to the two long cylindrical conductors of an overhead single-phase transmission line the diameter of each of which is comparable with the distance between the axes of the cylinders.*

ANSWER.—Let d cms. be the diameter of each cylinder, Fig. 32, and let a cms. be the distance between their axes.

It is first noted that the surfaces of the cylinders are equipotential surfaces, and the equation for these circles may be written—

$$r_1 = A\,r,$$

where A is a constant, $M : M_1$ are the points from which the distances r and r_1 are measured, and $O : O_1$ are the centres of the circles. The distance between the centres of the cylinders is a cms., and the distance between the points $M : M_1$ is b cms. It is easy to show that

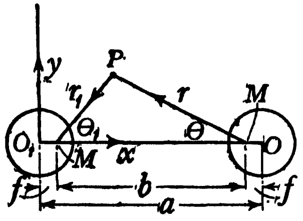


Fig. 32

$$b = \sqrt{a^2 - d^2},$$

d being the diameter of each cylinder (Fig. 32).

This may be proved as follows :—

Suppose O_1 is taken as the centre of Cartesian co-ordinates. Then for points on the circle whose centre is at O_1—

$$x^2 + y^2 - 2x\left[\frac{f - A^2(f+b)}{1 - A^2}\right] = \frac{A^2(f+b)^2 - f^2}{1 - A^2}.$$

But since the origin is at O_1, the coefficient of x must be zero—i.e.,

$$f = A^2(f+b),$$

or

$$A^2 = \frac{f}{f+b},$$

and the radius of the circle is

$$\frac{d}{2} = \sqrt{\frac{A^2(f+b)^2 - f^2}{1 - A^2}} = \sqrt{f^2 + fb}.$$

Therefore

$$f^2 + fb - \frac{d^2}{4} = o,$$

and

$$2f = (a - b).$$

Hence

$$\frac{(a-b)^2}{4} + \frac{(a-b)b}{2} - \frac{d^2}{4} = o\,;$$

or

$$b = \sqrt{a^2 - d^2}.$$

Now for equipotential lines of which the equation is—

$$r_1 = A\,r,$$

the corresponding lines of force will be given by the equation—

$$\theta_1 + \theta = \text{a constant},$$

as may be seen from page 92. These lines are circles passing through M and M_1. It follows, therefore, that the effect of two oppositely charged cylinders of diameter d, and whose axes are distant a apart, is the same as if the charges were concentrated on lines whose traces are shown at M and M_1 in Fig. 32, the distance between M and M_1 being

$$\sqrt{a^2 - d^2}.$$

From this it is seen that if d is small compared with a the points M and M_1 very nearly coincide with the centre points O and O_1 respectively, as was assumed to be the case previously.

The capacities of the cylinders for the case when $\frac{d}{a}$ is not small may now be deduced by the aid of these results.

The force at P (Fig. 33) is $\dfrac{2q}{\varepsilon r} + \dfrac{2q}{\varepsilon(b - r)}$ dynes.

The work done in moving unit charge from the surface of one conductor to the surface of the other is

$$\int_{r=\frac{d}{2}-f}^{r=b-\left(\frac{d}{2}-f\right)} \frac{1}{\varepsilon}\left[\frac{2q\,dr}{r}+\frac{2q\,dr}{b-r}\right]$$

$$=\frac{2q}{\varepsilon}\left[\log_e\left(b-\frac{d}{2}+f\right)^2-\log_e\left(\frac{d}{2}-f\right)^2\right]$$

$$=\frac{4q}{\varepsilon}\log_e\frac{b-\frac{d}{2}+f}{\frac{d}{2}-f}$$

$$=\frac{4q}{\varepsilon}\log_e\left[\frac{\sqrt{a^2-d^2}-(d-a)}{\sqrt{a^2-d^2}+(d-a)}\right],$$

since $b^2=a^2-d^2$, and $2f=a-b$

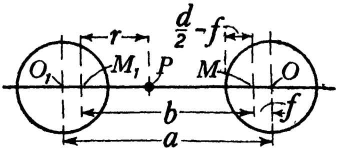


Fig. 33

Hence the capacity per cm. length of cable is—

$$C=\frac{q}{V}=\frac{\varepsilon}{4\log\left[\frac{\sqrt{a^2-d^2}-(d-a)}{\sqrt{a^2-d^2}+(d-a)}\right]},$$

or

$$C=\frac{\varepsilon}{4\log_e\left[\frac{\sqrt{a^2-d^2}+a}{d}\right]}$$ electrostatic c.g.s. units per cm. length

If the logarithm base be changed from e to 10, and the capacity expressed in practical units, then

$$C=\frac{\varepsilon}{51{\cdot}4\log_{10}\left[\frac{\sqrt{a^2-d^2}+a}{d}\right]}$$ microfarads per mile.

If $\frac{d}{a}$ is small this formula reduces to that given in expression 9, page 103, viz.,

$$C=\frac{\varepsilon}{82{\cdot}6\log_{10}\frac{2a}{d}}\ \mu\text{F per km.}$$

Chapter V

CURRENT DISTRIBUTORS AND NETWORKS

1.—A chain of two resistance cells is shown in the accompanying diagram. Find the resistance as measured between the terminals XY.

ANSWER.—*Method* (*i*). In Fig. 1 the arrow shows the direction in which the resistance calculation is worked back to the input terminals XY. The numbers shown on the diagram give the resistance in ohms of the individual links. Then :

Resistance just behind $BB = 1 + 4 = 5\,\Omega$
Conductance just behind $BB = \frac{1}{5} = 0{\cdot}2$ siemens.
Conductance just in front of $BB = \frac{1}{2} + 0{\cdot}2 = 0{\cdot}7$ siemens.

Resistance just in front of $BB = \dfrac{1}{0{\cdot}7} = 1{\cdot}43\Omega.$

Resistance just behind $CC = 2 + 1{\cdot}43 = 3{\cdot}43\,\Omega.$

Conductance just behind $CC = \dfrac{1}{3{\cdot}43} = 0{\cdot}29$ siemens.

Conductance at $XY = \frac{1}{2} + 0{\cdot}29 = 0{\cdot}79$ siemens.

Resistance at $XY = \dfrac{1}{0{\cdot}79} = \underline{1{\cdot}26\Omega.}$

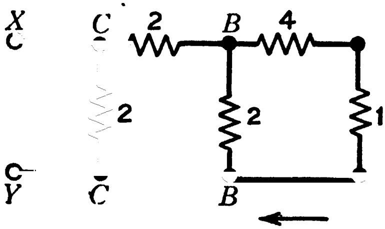


Fig. 1

Method (*ii*). In Fig. 2 it is assumed that an e.m.f. E is applied to the terminals XY by means of leads without resistance. Writing down the equations for the three cells I, II, III defining the total e.m.f. round a closed circuit to be zero, then

$$\left.\begin{aligned} &\text{I} \quad E = i_1 2 - i_2 2 + 0 \\ &\text{II} \quad 0 = -i_1 2 + i_2 6 - i_3 2 \\ &\text{III} \quad 0 = 0 - i_2 2 + i_3 7 \end{aligned}\right\}$$

The determinants are

$$D_1 = \begin{vmatrix} E & -2 & 0 \\ U & 6 & -2 \\ 0 & -2 & +7 \end{vmatrix} = E(42 - 4) = 38E$$

$$D = \begin{vmatrix} 2 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 7 \\ \\ 2 & -2 & 0 \end{vmatrix} \begin{matrix} = 2(42 - 4) - 2 \times 14 \\ = 48 \end{matrix}$$

Then $i_1 = \frac{E}{R_{xy}} = \frac{D^1}{D} = \frac{38E}{48}$ so that

$R_{xy} = \frac{48}{38} = \underline{1{\cdot}26\Omega}$ where R_{xy} is the resistance of the chain measured at the terminals X and Y.

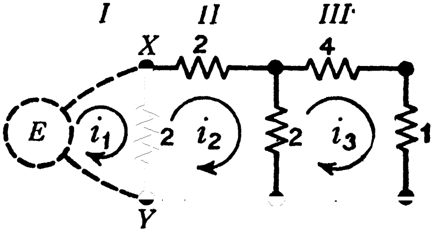


Fig. 2

Method (iii). The three simultaneous equations given in method (ii) can be solved directly, viz.,

$$E = 2i_1 - 2i_2$$
$$0 = -2i_1 + 6i_2 - 2i_3$$
$$0 = 0 - 2i_2 + 7i_3$$

Substituting $i_3 = \frac{2}{7} i_2$ gives

$$2i_1 = 6i_2 - \frac{4}{7} i_2 = i_2 \left(\frac{38}{7} \right) \text{ so that}$$

$$i_2 = \frac{14}{38} i_1 \text{ and } E = 2i_1 - \frac{28}{38} i_1$$

$$\text{or } E = i_1 \left(\frac{76 - 28}{38} \right) = \frac{48}{38} i_1$$

$$\text{that is, } R_{xy} = \frac{E}{i_1} = \underline{1{\cdot}26\Omega}$$

2.—A chain of three resistance cells is shown in Fig. 3. Find the resistance measured between the terminals XY.

Answer.—*Method* (*i*) : In Fig. 3 the arrow shows the direction in which the resistance is worked back to the input terminals $X\ Y$. Then

Resistance just behind AA $= 9\Omega$

Conductance just behind AA $= \frac{1}{9}$ siemens

Conductance just in front of AA $= 1 + \frac{1}{9} = \frac{10}{9}$ siemens

Resistance just in front of AA $= \frac{9}{10}\,\Omega$

Resistance just behind BB $= \frac{9}{10} + 4 = \frac{49}{10}\,\Omega$

Conductance just behind BB $= \frac{10}{49}$ siemens

Conductance just in front of BB $= \frac{1}{2} + \frac{10}{49} = \frac{69}{98}$ siemens

Resistance just in front of BB $= \frac{98}{69}\,\Omega$

Resistance just behind CC $= \frac{98}{69} + 2 = \frac{236}{69}\,\Omega$

Conductance just behind CC $= \frac{69}{236}$ siemens

Conductance at terminals XY $= \frac{1}{2} + \frac{69}{236} = \frac{187}{236}$ siemens

Resistance at terminals XY $= \frac{236}{187} = \underline{1{\cdot}261\ \Omega}$

Method (*ii*) : Again assuming an e.m.f. applied to the terminals XY by means of leads of no resistance, as shown in Fig. 4.

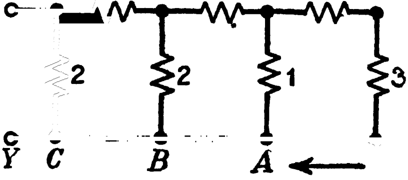


Fig. 3

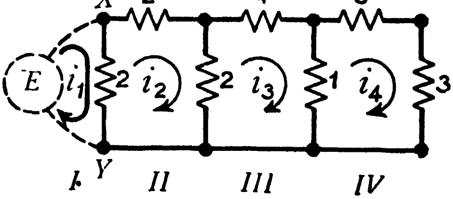


Fig. 4

Resistance
Cell No.

$$\begin{array}{ll} \text{I}: & E = 2\,i_1 - 2\,i_2 \\ \text{II}: & 0 = -2\,i_1 + 6\,i_2 - 2\,i_3 \\ \text{III}: & 0 = 0 - 2\,i_2 + 7\,i_3 - i_4 \\ \text{IV}: & 0 = 0 + 0 - i_3 \times 1 + 10\,i_4 \end{array}$$

Then substituting : $i_4 = \frac{1}{10} i_3$ gives

$$2i_2 = 7i_3 - \frac{1}{10} i_3 = 6{\cdot}9\, i_3$$

$$i_3 = \frac{2}{6{\cdot}9} i_2$$

$$2i_1 = 6i_2 - \frac{4}{6{\cdot}9} i_2 = i_2 \left(\frac{41{\cdot}4 - 4}{6{\cdot}9}\right) = \frac{37{\cdot}4}{6{\cdot}9} i_2$$

or

$$i_2 = \frac{13{\cdot}8}{37{\cdot}4} i_1$$

and hence

$$E = 2i_1 - \frac{13{\cdot}8}{18{\cdot}7} i_1 = i_1 \left(\frac{37{\cdot}4 - 13{\cdot}8}{18{\cdot}7}\right) = i_1 \frac{23{\cdot}6}{18{\cdot}7}$$

so that

$$\underline{R_{xy}} = \frac{23{\cdot}6}{18{\cdot}7} = \underline{1{\cdot}261\Omega}$$

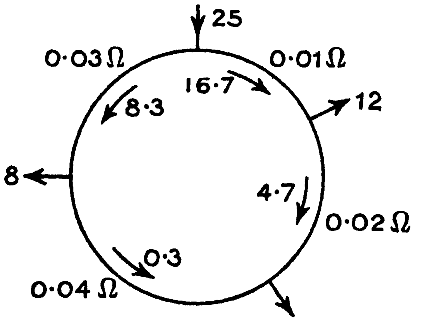


Fig. 5

3.—A ring-main is fed at one point and there are three consumer's tapping points, as shown in Fig. 5. Find the current distribution in the ring.

Answer.—*Method* (*i*) : The ring main is assumed to be cut at the feeding point and opened out as shown in Fig. 6, so that the problem becomes that of feeding a line from both ends which are at the same potential.

Then : $I' \times 0{\cdot}1 = 8 \times 0{\cdot}03 + 5 \times 0{\cdot}07 + 12 \times 0{\cdot}09$
$= 0{\cdot}24 + 0{\cdot}35 + 1{\cdot}08 = 1{\cdot}67$
$I' = 16{\cdot}7$ amperes : $I'' = 25 - 16{\cdot}7 = 8{\cdot}3$ amperes.

Method (*ii*) : Assume that each separate consumer is alone being supplied : find the corresponding current distribution, and superpose the results for all three consumers.

In Fig. 7*a* the 12 ampere tapping point is alone supposed to be operating.

Then $\frac{i_1}{i_2} = \frac{0{\cdot}09}{0{\cdot}01}, i_1 + i_2 = 12$, so that

$i_1 = 10{\cdot}8$ amperes : $i_2 = 1{\cdot}2$ amperes.

Similarly, from Fig. 7*b*: $\frac{i_1}{i_2} = \frac{0{\cdot}07}{0{\cdot}03} = \frac{7}{3}$

and $i_1 + i_2 = 5$, so that

$$i_1 + \frac{3}{7} i_1 = 5 \ : \ i_1 = \frac{7}{10} \times 5 = 3{\cdot}5 \text{ amperes}$$

$$i_2 = 1{\cdot}5 \text{ amperes}$$

and from Fig. 7*c*:

$$\frac{i_1}{i_2} = \frac{0{\cdot}03}{0{\cdot}07} = \frac{3}{7}$$

$$i_1 + i_2 = 8 \ : \ i_1 + \frac{7}{3} i_1 = 8$$

$$\frac{10}{3} i_1 = 8 \ : \ i_1 = 2{\cdot}4 \text{ amperes}$$

$$i_2 = 5{\cdot}6 \text{ amperes}$$

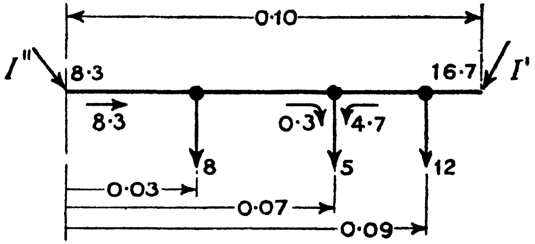
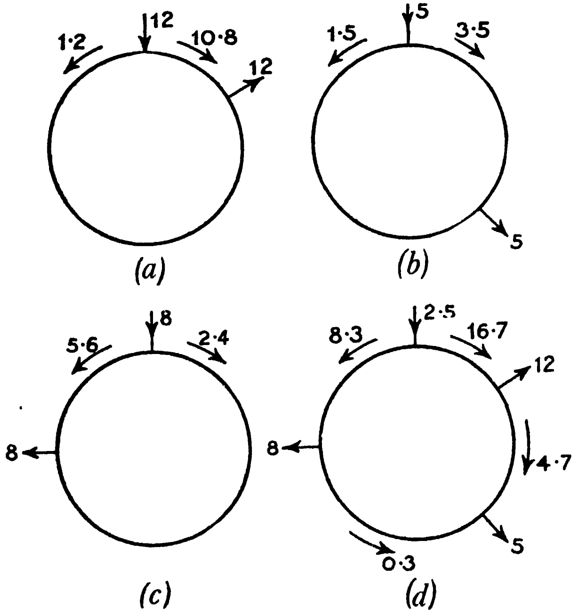


Fig. 6

Superposing the three results of Figs. 7*a*, 7*b* and 7*c*, it is seen that of the 25 amperes supplied to the ring at the feeding point

10·8 + 3·5 + 2·4 = 16·7 amperes goes in the direction towards the tapping point of 12 amperes and 8·3 amperes in the other direction as shown in Fig. 7*d*.

Since the tapping point of 5 amperes is seen to be fed from both sides, this is the point of maximum pressure drop.



(*a*) (*b*) (*c*) (*d*)

Fig. 7

4.—A closed ring-main, shown in Fig. 8, is fed at one point A and supplies three separate consumers. Find the current distribution by assuming the main is cut at any point, such as B.

ANSWER.—The ring-main is assumed to be cut at the point B and opened out as shown in Fig. 9. The equation can then be written down expressing that the potential at the two ends of the line is the same, viz.,

$$0{\cdot}003X = 0{\cdot}002(390 - X) + 0{\cdot}004(240 - X) + 0{\cdot}004(130 - X)$$

where X amperes is the current which flows in the section ACB in Fig. 8.

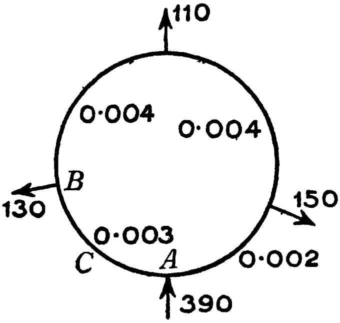


Fig. 8

This equation reduces to

$$0{\cdot}003X = 2{\cdot}26 - 0{\cdot}01X$$

so that $X = 175$ amperes

and the current distribution is shown in Fig. 9.

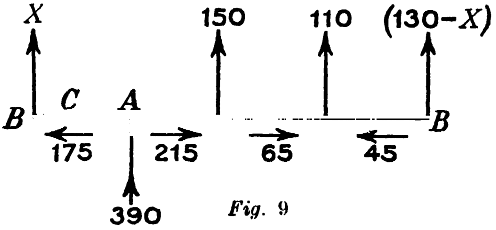


Fig. 9

5.—A star connected system of resistance is loaded, as shown in Fig. 10, and the supply points I, II and III are all at the same potential. Find (i) the current distribution in the system, and (ii) the potential difference between the star point and the supply points.

ANSWER.—Method (*i*) : Assume that two of the star limbs in Fig. 10 are each cut through at points indefinitely near the star point O, as shown in Fig. 11. Suppose the current which is crossing at A before cutting is x amperes and at B it is y amperes. Then :

For the circuit IAII :

$$[60 - (x + y)]0{\cdot}3 - (x + y)0{\cdot}1 = (20 + x)0{\cdot}2 + x0{\cdot}3$$

that is

$$9x + 4y = 140$$

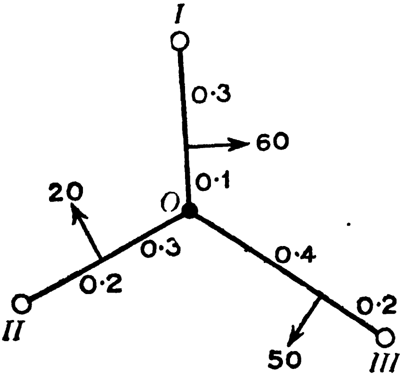


Fig. 10

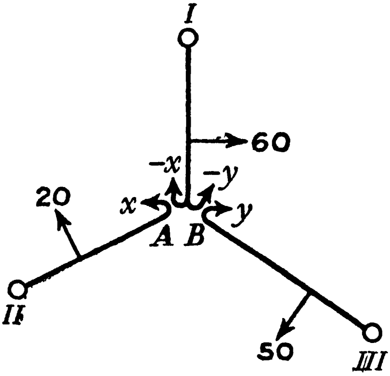


Fig. 11

Similarly, for the circuit I*B*III :

$$(50 + y)0{\cdot}2 + 0{\cdot}4y = [60 - (x + y)]0{\cdot}3 - (x + y)0{\cdot}1$$

that is

$$10y + 4x = 80$$

From the two equations for x and y it is found that

$$x = 14{\cdot}6 \text{ amperes} : y = 2{\cdot}15 \text{ amperes} : x + y = 16{\cdot}75 \text{ amperes.}$$

The current distribution in the system is then as shown in Fig. 12.

The pressure drop from $\text{I} - \text{O} = 43{\cdot}25 \times 0{\cdot}3 = 16{\cdot}75 \times 0{\cdot}1$
$= 11{\cdot}3$ volts

The pressure drop from $\text{II} - \text{O} = 34{\cdot}6 \times 0{\cdot}2 + 14{\cdot}6 \times 0{\cdot}3$
$= 11{\cdot}3$ volts

The pressure drop from $\text{III} - \text{O} = 52{\cdot}15 \times 0{\cdot}2 + 2{\cdot}15 \times 0{\cdot}4$
$= 11{\cdot}3$ volts

Method (*ii*) : The current in each limb of the star connection is referred to the corresponding supply point and the star point O Fig. 13.

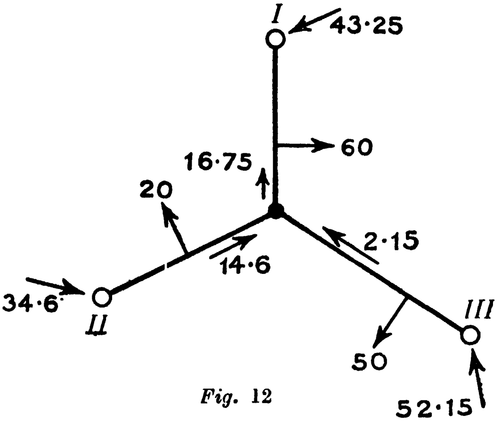


Fig. 12

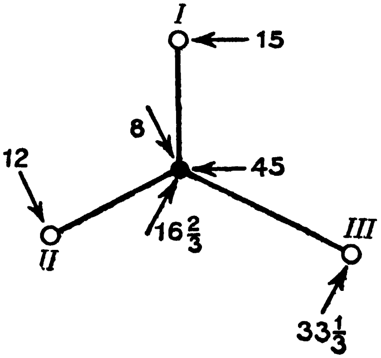


Fig. 13

Thus, for limb I O :

$$\text{I}' = \frac{60 \times 0{\cdot}1}{0{\cdot}4} = 15 \text{ amperes} : \text{I}'' = 45 \text{ amperes}$$

For the limb II O :

$$\text{I}' = \frac{20 \times 0{\cdot}3}{0{\cdot}5} = 12 \text{ amperes} : \text{I}'' = 8 \text{ amperes}$$

For the limb III O:

$$\text{I}' = \frac{50 \times 0{\cdot}4}{0{\cdot}6} = 33\tfrac{1}{3} \text{ amperes} : \text{I}'' = 16\tfrac{2}{3} \text{ amperes}$$

and those component currents are shown in Fig. 13, from which it is seen that the total current at the star point = 69⅔ amperes, and this would be the current which would have to be fed in at the star point O if that point were at the same potential as the supply points I, II and III. But no current is actually fed in at the star point, so that the current of 69⅔ amperes must pass through the parallel connection of the three star limbs as shown in Fig. 14.

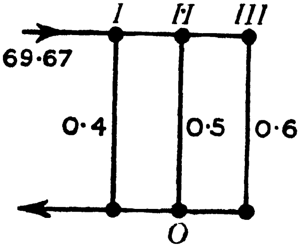


Fig. 14

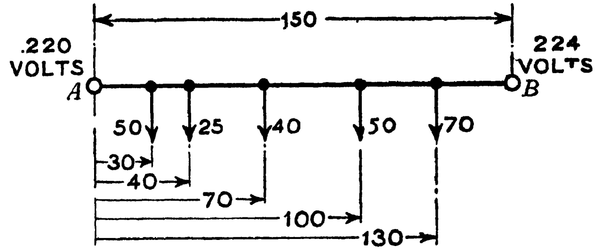


Fig. 15

The resistance of the parallel combination of Fig. 14 is

$$R_0 = \frac{1}{\frac{1}{0{\cdot}4} + \frac{1}{0{\cdot}5} + \frac{1}{0{\cdot}6}} = \frac{1}{2{\cdot}5 + 2{\cdot}0 + 1{\cdot}67}$$

$$= \frac{1}{6{\cdot}17} = 0{\cdot}162\Omega.$$

The pressure drop due to the passage of 69⅔ amperes through this resistance is

$$69{\cdot}67 \times 0{\cdot}162 = 11{\cdot}3 \text{ volts.}$$

The current distribution in the star is then easily found thus (see Fig. 10),

For limb I O: $i \times 0{\cdot}3 + (i - 60) \times 0{\cdot}1 = 11{\cdot}3$ volts.

so that $i = 43{\cdot}25$ amperes

For limb II O: $i \times 0{\cdot}2 + (i - 20)\,0{\cdot}3 = 11{\cdot}3$ volts.

so that $i = 34{\cdot}6$ amperes

For limb III O: $i \times 0{\cdot}2 + (i - 50)\,0{\cdot}4 = 11{\cdot}3$ volts.

so that $i = 52{\cdot}15$ amperes

thus being in accord with the results obtained by method (*i*).

6.—A distributor of copper wire 150 metres long, as shown in Fig. 15, is fed from each of the ends A and B, the end A being supplied at 220 volts and the end B at 224 volts. Loads are connected to tapping points as follows (distances measured from the end A):

50 amperes at 30 metres; 25 amperes at 40 metres; 40 amperes at 70 metres; 50 amperes at 100 metres; 70 amperes at 130 metres. Find what current is fed in at each end of the line and the maximum percentage drop. The cross section of the distributor conductor is 22 sq. mm.

ANSWER.—(*i*) The currents which would have to be fed to the line from the two ends *A* and *B* if their two feeding points were at the same potential would be (Fig. 15):

$$\textit{End } B:\ \mathrm{I}'' = \frac{70 \times 130 + 50 \times 100 + 40 \times 70 + 25 \times 40 + 50 \times 30}{150}$$

$$= 129 \text{ amperes.}$$

$$\textit{End } A:\ \mathrm{I}' = \left[(\text{Total current taken}) - 129\right] \text{ amperes}$$

$$= 235 - 129 = \underline{106 \text{ amperes.}}$$

$$\text{Resistance of the distributor} = \frac{150}{\lambda \times q} = \frac{150}{56 \times 22} = 0{\cdot}12\Omega.$$

$$\text{Resistance per metre} \doteq \underline{0{\cdot}0008\Omega}.$$

(*ii*) The compensating current flowing from end *B* to end *A* and due to the p.d. of 4 volts is

$$\mathrm{I}c = \frac{4}{0.12} = \underline{33.3 \text{ amperes}}$$

so that the current fed in at end *B* = 162·3 amperes

so that the current fed in at end *A* = 72·7 amperes

giving the current distribution shown in Fig. 16, from which it is seen that the maximum pressure drop occurs at the 25 ampere tapping point. The pressure drop at this tapping point referred to the potential of 220 volts at end *A* is

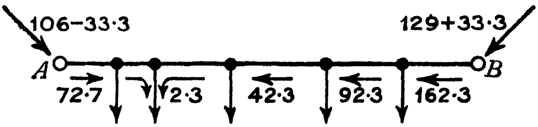


Fig. 16

$$\triangle v = 30 \times 0{\cdot}0008 \times 73 + 23 \times 0{\cdot}0008 \times 10 = 1{\cdot}75 + 0{\cdot}184 = 1{\cdot}934\, v.$$

and for "go" and "return": drop $= 2\triangle v = 3{\cdot}868$ volts.

$$\text{Maximum percentage pressure drop is then } \frac{3{\cdot}87}{220} \times 100 = \underline{1.85 \text{ per cent.}}$$

7.—Find by the method of determinants the current distribution in the network shown in the accompanying diagram.

ANSWER.—The three simultaneous equations for the network of Fig. 17 are:

$$\begin{aligned} E_1 &= I_1 2R - I_2 R \\ E_2 &= I_2 3R - I_1 R - I_3 R \\ E_3 &= I_3 2R - I_2 R \end{aligned}$$

that is,

$$\begin{aligned} E_1 &= I_1 2R + I_2(-R) + O \\ E_2 &= I_1(-R) + I_2 3R + I_3(-R) \\ E_3 &= O + I_2(-R) + I_3 2R \end{aligned}$$

so that the resistance determinant is

$$D = \begin{vmatrix} 2R & -R & O \\ -R & 3R & -R \\ O & -R & 2R \end{vmatrix} = 2R(6R^2 - R^2) - R(2R^2)$$

$$\begin{array}{ccc} -2R & -R & O \\ -R & 3R & -R \end{array}$$

hence

$$\underline{D = 8R^3}$$

Then $i_1 = \frac{D_1}{D}$ where D_1 is obtained by replacing column 1 of D by the respective e.m.f's. $E_1 : E_2 : E_3$ thus:

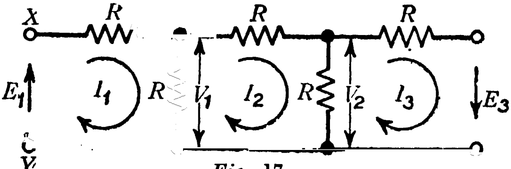


Fig. 17

$$I_1 = \frac{\begin{vmatrix} E_1 & -R & O \\ E_2 & 3R & -R \\ E_3 & -R & 2R \end{vmatrix}}{D} = \frac{E_1(6R^2 - R^2) + E_2 2R^2 + E_3 R^2}{D}$$

and, putting $E_2 = O : E_3 = \frac{1}{2}E_1$ then

$$\underline{I_1 = \frac{E_1 5R^2 + \frac{1}{2}E_1 R^2}{8R^3} = \frac{11}{16}\frac{E_1}{R}}$$

Similarly, $I_2 = \frac{D_2}{D}$ where

$$D_2 = \begin{vmatrix} 2R & E_1 & O \\ -R & E_2 & -R \\ O & E_3 & 2R \end{vmatrix} = 2R(2E_2R + E_3R) - R(-2E_1R)$$

$$\begin{array}{ccc} 2R & E_1 & O \\ -R & E_2 & -R \end{array}$$

and, as before, putting $E_2 = O : E_3 = \frac{1}{2}E_1$

$$D_2 = 2E_3R^2 + 2E_1R^2 = 3E_1R^2$$

and

$$\underline{I_2 = \frac{D_2}{D} = \frac{3}{8}\frac{E_1}{R}}$$

Also:

$$I_3 = \frac{D_3}{D} = \frac{\begin{vmatrix} 2R & -R & E_1 \\ -R & 3R & E_2 \\ O & -R & E_3 \end{vmatrix}}{D} = \frac{[2R^2(3E_3 + E_2') - R^2(-E_1 + E_3)]}{D}$$

and for $E_2 = O : E_3 = \frac{1}{2}E_1$

$$\underline{I_3 = \frac{3{\cdot}5E_1R^2}{D} = \frac{7}{16}\frac{E_1}{R}}$$

A check on these results can be obtained as follows:

$$V_1 = E_1 - I_1R = E_1 - \frac{11}{16}E_1 = \frac{5}{16}E_1$$

$$\begin{cases} V_2 = V_1 - I_2R = \frac{5}{16}E_1 - \frac{3}{8}E_1 = -\frac{1}{16}E_1 \\ \text{also,} \\ V_2 = I_2R - I_3R = \frac{3}{8}E_1 - \frac{7}{16}E_1 = -\frac{1}{16}E_1 \end{cases}$$

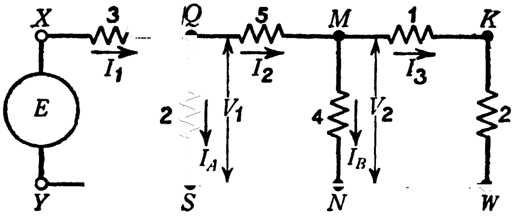


Fig. 18

8. — Apply Thévenin's Theorem to determine the current distribution shown in the network of Fig. 18.

ANSWER.—(i) Assume the link *QS* in Fig. 18 has been removed and find the p.d. e_0 across *Q* and *S* as shown in Fig. 19.

The resistance of the cell *MKWN* is

$$\frac{1}{\frac{1}{4} + \frac{1}{3}} = \frac{12}{7} = 1{\cdot}714\Omega$$

so that the resistance between the terminals *XY* in Fig. 19 is 9·714Ω. The current is $I = \frac{E}{9{\cdot}714}$ amperes, so that the pressure across *Q* and *S* is

$$e_0 = E - I \times 3 = E - \frac{3E}{9{\cdot}714} = 0{\cdot}692E.$$

(ii) Assume the points *Q* and *S* are short-circuited as shown in Fig. 20 then the short-circuit current is

$$i_k = \frac{E}{3}$$

and the short-circuit resistance as measured from *QS* is

$$r_k = \frac{e_o}{i_k} = 0\cdot692 \times 3 = 2\cdot076\Omega$$

Hence the current in the link *QS* when replaced in position will be:

$$I_A = \frac{e_o}{R_{QS} + 2\cdot076} = \frac{0\cdot692}{4\cdot076} E = \underline{0\cdot17\,E \text{ amperes.}}$$

The currents in the rest of the system are then at once obtainable. Thus

$$3I_1 = E - 0\cdot17E \times 2 = E \times 0\cdot66$$

$$\underline{I_1} = \underline{0\cdot22\,E \text{ amperes}}$$

$$I_2 = I_1 - I_A = (0\cdot22 - 0\cdot17)E = \underline{0\cdot05E \text{ amperes}}$$

$$V_2 = I_B \times 4 = V_1 - 5I_2 = 2I_A - 5I_2$$

$$\therefore 4I_B = (0\cdot34 - 0\cdot25)E = 0\cdot09F$$

$$\underline{I_B = 0\cdot0225E \text{ amperes}}$$

$$I_3 = I_2 - I_B = (0\cdot05 - 0\cdot0225)E = \underline{0\cdot0275E \text{ amperes}}$$

These results should be confirmed by solving the problem by means of the full set of simultaneous equations for the system.

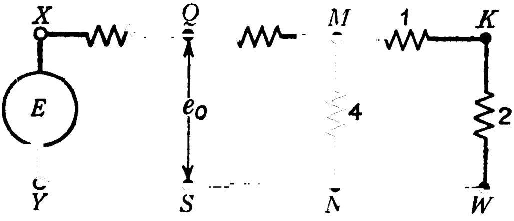


Fig. 19

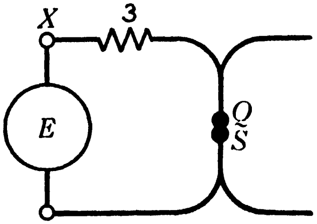


Fig. 20

9.—*Give an account of (i) the Varley test and (ii) the Murray test for determining the position of faults in cables.*

ANSWER.—These tests are Wheatstone Bridge type of d.c. methods for measuring the position of a fault due to insulation breakdown in a cable. They are usually applied when faults in light-current cables are to be located, because in such multiple core cables it is generally possible to find a sound conductor to use as the return lead and also there are frequently neighbouring cables available for the purpose.

When the fault is near the place at which the test is set up, the Varley method is not sufficiently sensitive, in which case the Murray test is preferable. If, however, the Varley test is to be used here also, it is desirable to use a slide wire resistance in place of the Bridge resistance.

(i) *The Varley Test : Fig.* 21. For this test it is necessary to have a sound line by means of which the testing bridge can be connected to a point beyond the fault in the defective cable as shown in Fig. 21. The resistance of the loop $(l + m)$ is first measured by the normal Wheatstone Bridge connection and let this value be K ohms. The connections are then arranged as shown in Fig. 21, and when balance is obtained by adjusting Q, then

$$\frac{R}{S} = \frac{Q + x}{K - x}$$

If R is made equal to S, then

$$Q + x \doteq K - x$$

$$x = \frac{K - Q}{2} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

and, knowing the resistance per mile of the cable the position of the fault is obtained.

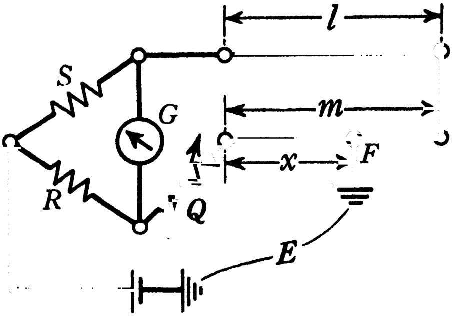


Fig. 21

(ii) *The Murray Test : Fig.* 22. In this arrangement of the Bridge connections when balance is obtained

$$\frac{S}{R} = \frac{x}{K - x}$$

so that $x = \dfrac{K \times S}{R + S}$ .... (ii)

It is to be observed that these two tests are associated in that the Varley test is used to give the approximate position of the fault and the Murray test to locate its exact position more closely.

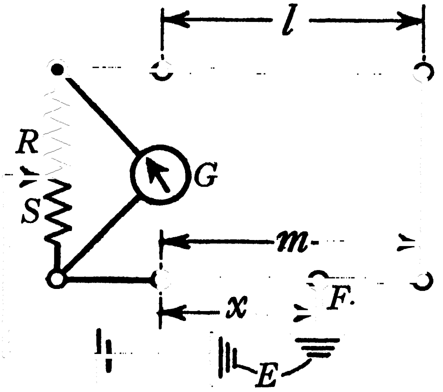


Fig. 22

Thus in the preliminary test the fault may have to be located in a long line of cables in series so that in expression (i) K and Q will both be relatively large, as well as the difference $K—Q$, so that small percentage errors in the measurement of K and Q will not give rise to large percentage errors in the measurement of x. When, however, the fault has been traced down to a single length of cable the expression (ii) for the Murray test does not involve the difference of two resistance measurements so that small percentage errors in the measurement of the resistances appear in the final expression in the same percentage.

10.—Describe the " universal shunt " and explain its purpose.

ANSWER.—When using a shunted galvanometer to compare the strengths of two currents, it is not necessary to know what is the exact fraction of the main currents which pass through the galvanometer. By means of the universal shunt the currents can be compared if it is known what fraction of the galvanometer shunt resistance is tapped off by the supply lead. This will be apparent from the following relationships, it being assumed that the galvanometer deflection is proportional to the galvanometer current.

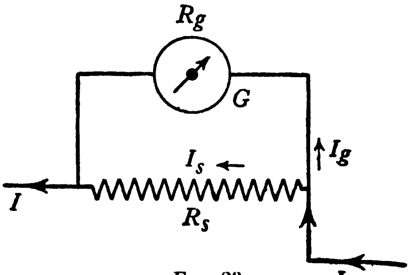


Fig. 23

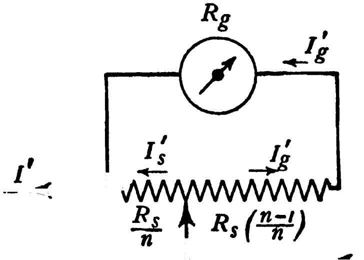


Fig. 24

In Fig. 23 a resistance $R_s\Omega$ is shown connected across the galvanometer terminals, the internal resistance of the galvanometer being $R_g\Omega$, so that,

$$\frac{I_g}{I_s}=\frac{R_s}{R_g}$$

and the main current is

$$I = I_g + I_s = I_g\left(1 + \frac{R_g}{R_s}\right)$$

that is
$$I = I_g\left(\frac{R_s + R_g}{R_s}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

In Fig. 24 the main current I' is supplied through a tapping point of the resistance R_s, so that the portion $R_s\left(\frac{n-1}{n}\right)$ becomes a series resistance to the galvanometer. and the portion $\frac{R_s}{n}$ is then in parallel with the resistance $R_g + R_s\left(\frac{n-1}{n}\right)$ of the galvanometer circuit. The following relationships will then hold, viz.

$$I' = I'_g\left[\frac{\frac{R_s}{n} + R_g + R_s\left(\frac{n-1}{n}\right)}{\frac{R_s}{n}}\right]$$

that is,
$$I' = I'\,n\left(\frac{R_s + R_g}{R}\right) \quad \text{(ii)}$$

Hence from equations (1) and (2)
$$\frac{I'}{I} = n\,\frac{I'_g}{I_g} = n\,\frac{\theta'}{\theta} \quad \text{(iii)}$$

where θ' and θ are the respective values of the galvanometer for the two conditions. Thus, if $n=100$, the ratio of the main currents will be
$$\frac{I'}{I} = 100\,\frac{\theta'}{\theta}$$

and is independent of the individual values of the resistances R_s and R_g. That is to say, a shunt resistance of $R_s\Omega$ with suitable tapping points can be used with any galvanometer of internal resistance $R_g\Omega$ and for this reason it was termed the " universal shunt " by Ayrton and Mather to whom the idea is due.

Chapter VI

THERMO-ELECTRICITY : PIEZO-ELECTRICITY

1.—When an electric current passes across a junction of two different metals the " Joule effect " and the " Peltier effect " are superposed. How may these two effects be separated ?

ANSWER.—In Fig. 1 is shown the relationship between e.m.f. and current for a copper-constantan couple from which it is seen that when the current is flowing in the direction across the junction from copper to constantan the curve rises steeply as the current increases. For this direction of current the Joule effect (I^2R) and the Peltier effect each produce a heating action so that the temperature rise is that due to the sum of the two effects. When, however, the current is flowing across the junction in the direction from constantan to copper, the Peltier effect produces a *cooling* action whilst the Joule effect is still a heating action, so that the actual temperature acquired by the junction is that due to the difference of the two effects. This is seen from the portion of the curve on the right-hand side of the ordinate axis in Fig. 1 for which the current was passed from constantan to copper across the junction. For low values of the current the e.m.f. developed is first negative—reaching a maximum negative value for a current of about 6 amperes—and then gradually falling until it becomes zero when the current is 12.2 amperes For higher current values the e.m.f. becomes positive.

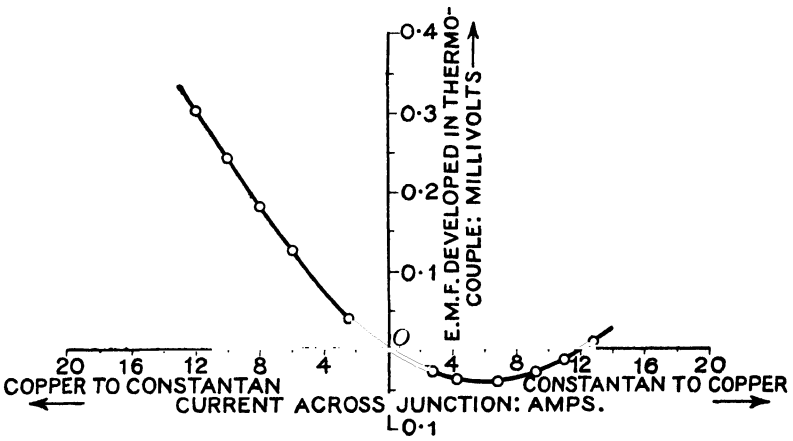


Fig. 1

It is seen, therefore, that for this couple structure the Peltier cooling effect just neutralizes the Joule heating effect when the current is about 12·2 amperes and this result provides a means for separating the two effects as follows :—Referring to Fig. 2, the two chain-dotted curves *A*, *C*, *B* and *G*, *K*, *H* are reproduced portions of the curve of Fig. 1. The ordinate *CF* is drawn to correspond to the current of 12·2 amperes, that is, the current corresponding to the point *K* of Fig. 2 where the curve *GKH* crosses the abscissa axis. This ordinate *CF* is bisected at *D* and the straight line *DO* is drawn through the origin. Then the ordinates of the straight line *DO* give the value of the Peltier effect and by subtracting the ordinates of *DO* from the corresponding ordinates of the curve in Fig. 1 the chain-dotted curve marked " Joule effect " in Fig. 2 is obtained.

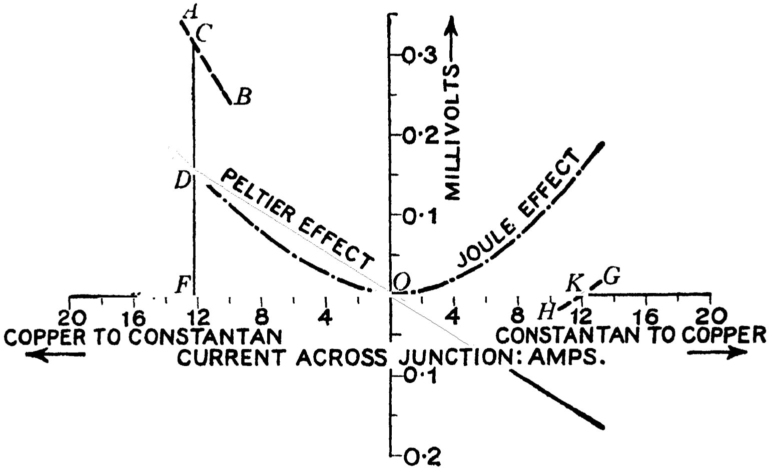


Fig. 2

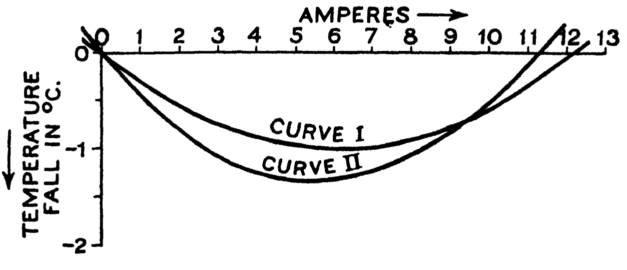


Fig. 3

In Fig. 3 are shown the values of the fall of temperature for a copper-constantan junction (curve I) and for an iron-constantan junction (curve II). (See : T. F. Wall " The Electrician," March 16th, 1928, p. 297.)

2.—Show how the Peltier cooling action may be calculated from the e.m.f./temperature relationship of a thermo-junction.

ANSWER.—It can be shown that the rate of heating or cooling of a junction of two dissimilar metals due to the passage of a current across the junction is given by the expression

$$T\frac{dV}{dT} \text{ joules per ampere per second.}$$

that is

$$0{\cdot}239T\frac{dV}{dT} \text{ calories per ampere per second.}$$

In this expression T is the temperature of the junction measured in absolute degrees, and $\frac{dV}{dT}$ is the slope at the temperature T of the curve connecting e.m.f. in volts and the temperature difference in deg. C. of the hot and cold junctions.

Referring to Fig. 4 it will be seen that for an iron-constantan junction at 15 deg. C.

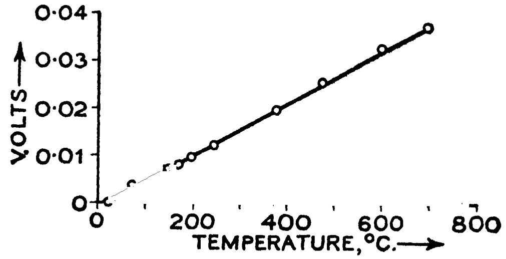


Fig. 4

$$\frac{dV}{dT} = 0{\cdot}055 \times 10^{-3} \text{ volt per 1 deg. C.}$$

Now, 15 deg. C. is equal to 288 deg. absolute, so that

$$T\frac{dV}{dT} = 0{\cdot}0158 \text{ joule per ampere per second, or,}$$

$$0{\cdot}0038 \text{ calorie per ampere per second.}$$

For the particular design of iron-constantan junction used it is seen by reference to Fig. 5 that when a current of 11·2 A is flowing from constantan to iron across the junction, the cooling due to the Peltier effect is just equal to the heating due to the Joule effect, that is,

$$I^2R = T\frac{dV}{dT}I,$$

where R ohms is the resistance of the junction.

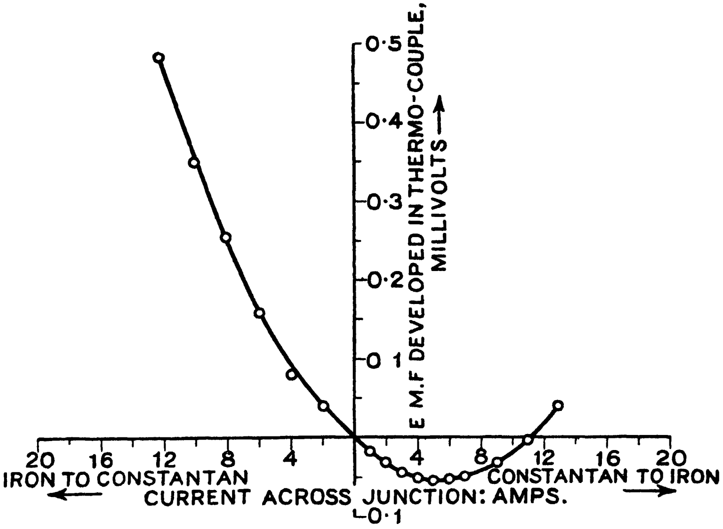


Fig. 5

Hence,

$$IR = T\frac{dV}{dT}$$

or,

$$R = \frac{T}{I}\frac{dV}{dT}$$

that is,

$$R = 0{\cdot}0014 \text{ ohm}.$$

It is also seen by reference to Fig. 6 that when a current of 5·6 A is flowing across the junction from constantan to iron the rate of absorption of heat due to the Peltier effect is twice the rate of generation of heat due to the Joule effect. That is to say, the net cooling effect on the junction due to a current of 5·6 A will be

$$\tfrac{1}{2}\, T\frac{dV}{dT}I \text{ joule per second,}$$

$$= \tfrac{1}{2} \times 0{\cdot}0158 \times 5{\cdot}6 \text{ joule per second,}$$

$$= 0{\cdot}0444 \text{ joule per second,}$$

that is,

$$\underline{0{\cdot}0106 \text{ calorie per second.}}$$

It is of interest to inquire what can be done by means of a cooling effect of this magnitude.

If a cooling effect of 0.01 calorie per second be allowed to act for one hour, the amount of heat absorbed would be 36 calories. That is to say, when a current of 5·6 A is passed across this particular junction for one hour it would have caused an absorption of heat sufficient to lower the temperature of one pint of water by 0·06 deg. C. Further, a single junction of iron-constantan of the design used would absorb sufficient heat to cause the temperature of one pint of air to fall in one minute by 5·5 deg. C., taking the specific heat of air at constant volume to be 0·17 and the density of air to be 840 c.c. per gm.

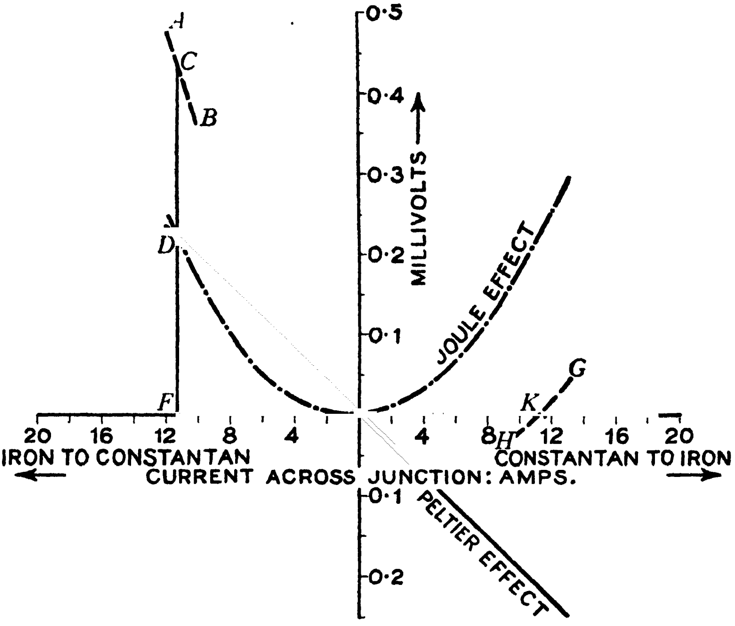


Fig. 6

(See : T. F. W. " The Electrician," March 16th, 1928, p. 297.)

3.—Quartz plates for piezo-electric oscillating systems are usually cut from the crystal, as shown in Fig. 7. If such a quartz plate is held between two metal plates so that the electric axis of the crystal lies at right angles to the metal plates, then, according to the frequency of the p.d. which is applied to the plates, longitudinal or transverse (i.e. thickness) oscillations of the quartz will be produced. What is the structure of the electric circuit which is equivalent to this system of quartz and metal plates ?

Answer.—The selection of the quartz plate from the crystal is shown in Fig. 7 and the arrangement of quartz and metal plates is shown diagrammatically in Fig. 8*a*, and in Fig. 8*b* is shown the equivalent electric circuit, the values of the circuit constants being given in Table I.

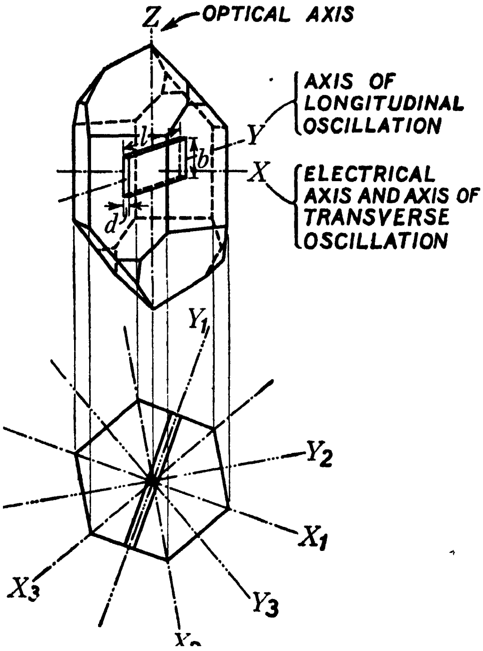

Z
OPTICAL AXIS
AXIS OF LONGITUDINAL OSCILLATION
Y
X
ELECTRICAL AXIS AND AXIS OF TRANSVERSE OSCILLATION
l
b
d
Y1
Y2
X1
X3
Y3

Fig. 7

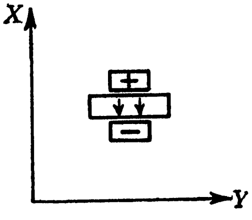


Fig. 8*a*

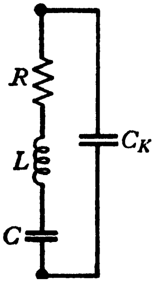


Fig. 8*b*

TABLE I

CIRCUIT CONSTANT	LONGITUDINAL OSCILLATIONS	TRANSVERSE OSCILLATIONS
R (OHMS)	$\frac{\pi^2 Q\rho}{4\varepsilon^2_{11}} \frac{d}{bl} 9 \times 10^{11}$	$\frac{\pi^2 Q\rho}{8\,\varepsilon^2_{11}} \frac{d}{bl} 9 \times 10^{11}$
L (HENRY)	$\frac{\rho}{8\,\varepsilon^2_{11}} \frac{ld}{b} 9 \times 10^{11}$	$\frac{\rho}{8\,\varepsilon^2_{11}} \frac{d^3}{bl} 9 \times 10^{11}$
C (FARAD)	$\frac{8\,\varepsilon^2_{11}}{\pi^2 E} \frac{bl}{d} \frac{1}{9 \times 10^{11}}$	$\frac{8\,\varepsilon^2_{11}}{\pi^2 E} \frac{bl}{d} \frac{1}{9 \times 10^{11}}$
C_k (FARAD)	$\frac{\varepsilon}{4\pi} \frac{bl}{d} \frac{1}{9 \times 10^{11}}$	$\frac{\varepsilon}{4\pi} \frac{bl}{d} \frac{1}{9 \times 10^{11}}$

where:

l *cm.* = length of the crystal.
b *cm.* = breadth of the crystal.
d *cm.* = thickness of the crystal.
$Q = 107{\cdot}5$ = coefficient of internal friction of the crystal.
$\rho = 2{,}654$ gm. per c.cm. is the density of the crystal.
$\varepsilon_{11} = -4{\cdot}77 \times 10^4$ = piezo-electric constant.
$E = 8{\cdot}51 \times 10^{11}$ dynes/cm.^{2} is Young's modulus for the crystal.
$\varepsilon = 4{\cdot}55$ is the dielectric constant of the crystal.

If these numerical values are inserted for the constants in TABLE I, the results given in TABLE II will be obtained.

TABLE II

CIRCUIT CONSTANT	LONGITUDINAL OSCILLATIONS	TRANSVERSE OSCILLATIONS
R (OHMS)	$2{\cdot}79 \times 10^5 \frac{d}{bl}$	$1{\cdot}395 \times 10^5 \frac{d}{bl}$
L (HENRY)	$131 \frac{ld}{b}$	$131 \frac{d^3}{lb}$
C (FARAD)	$2{\cdot}405 \times 10^{-15} \frac{bl}{d}$	$2{\cdot}405 \times 10^{-15} \frac{bl}{d}$
C_k (FARAD)	$4{\cdot}02 \times 10^{-13} \frac{bl}{d}$	$4{\cdot}02 \times 10^{-3} \frac{bl}{d}$

The individual numerical values may vary somewhat according to the quality of the quartz crystal.

The impedance $\mathfrak{Z}$ of the crystal is mainly defined by the constants R, L, and C, viz.,

$$\mathfrak{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)\Omega$$

For resonance
$$f_o \simeq \frac{1}{2\pi\sqrt{LC}}\, Hz$$

and, inserting the numerical values for L and C,

$$\begin{cases} f_o = \frac{k}{2l}\sqrt{\frac{E}{\rho}} = \frac{k}{l}\, 2{\cdot}835 \times 10^5\ Hz \text{ for longitudinal oscillations.} \\ f_o = \frac{k}{2d}\sqrt{\frac{E}{\rho}} = \frac{k}{d}\, 2{\cdot}835 \times 10^5\ Hz \text{ for transverse oscillations.} \end{cases}$$

$$\begin{cases} \lambda = \frac{c}{f_o} = 3 \times 10^8 \frac{l}{k\, 2{\cdot}835 \times 10^5} = 1060 \frac{l}{k} \text{ metres for longitudinal oscillations.} \\ \lambda = 1{,}060 \frac{d}{k} \text{ metres for transverse oscillations} \end{cases}$$

The factor $k = 1, 2, 3, \ldots$ is the order of the oscillations since, in addition to the fundamental frequency ($k = 1$) a series of higher harmonics can be produced. For the even harmonics, however, multi-electrode arrangements are usually necessary.

The damping factor $\delta = \frac{R}{2L}$ and the logarithmic decrement of the crystal $\triangle = \pi R \sqrt{\frac{C}{L}}$ are respectively:

$$\delta = \frac{1065}{l^2} : \triangle = 3{\cdot}75 \times 10^{-3} \frac{1}{l} \text{ for longitudinal oscillations.}$$

$$\delta = \frac{1065}{d^2} : \triangle = 3{\cdot}75 \times 10^{-3} \frac{1}{d} \text{ for transverse oscillations.}$$

4.—Give a short account of the problem of measuring high temperatures such as are required for many industrial purposes.

ANSWER.—Methods for the measurement of high temperatures within the technically important range round about 1,000° C. are required for many technical processes such as the melting, annealing, and hardening of steel. At present, such methods are almost entirely confined to the use of Seger cones, thermo-couples, and radiation pyrometers. These appliances are fundamentally unsuitable for the direct automatic recording and regulation of furnace temperatures, since, for example, in the case of the thermo-couple method, the small magnitude of the electrical quantities involved requires sensitive amplifying apparatus.

In order to meet the multitude of varieties of industrial requirements, a whole series of instruments for temperature control have been developed in recent years and the various problems involved are being energetically investigated.

In one case, for example, it may be necessary to maintain the temperature of an incubator constant to within a fraction of 1°; in another case it may be required to indicate the occurrence of excessive temperature rises in the pipes of a heating system without having to measure such temperatures; in yet another case it may be necessary to measure accurately and to record the steam temperature at the inlet pipe of a steam turbine. Another type of problem is to maintain a furnace at constant temperature by the accurate dosage of energy supply, or to vary the temperature according to a process chart, with or without continuous recording.

For platinum thermo-couples, as are commonly used for temperatures above 1,000° C., refractory protective tubes have been developed which lengthen both the life of the thermo-couple and the suitable intervals between calibrations of the couple. In practice, a gas-tight protective tube is essential to prevent any deleterious action on the couple, and only *ceramic materials can be used* for the high temperatures which are involved. Good protection is given by *porcelain such as "pythagoras" porcelain.* Sinter-corund tubes are gas-tight for temperatures far beyond that which can be regarded as a safe limit for a platinum thermo-couple.

A new method for measuring surface temperatures has been developed which comprises the formation (by means of a lens) of a view of an extended heat-radiating surface on a photographic plate. This method depends upon the action of the electromagnetic waves radiated from a hot surface and caught by a photographic plate which is made sensitive to infra-red rays. Different temperatures produce different degrees of blackening of such plates, and in this way a clearly defined chart of the temperature distribution and of the equi-temperature lines is obtained. Such measurement can be made with about the same degree of accuracy as that which can be obtained by the use of thermo-couples, and a notable feature of the method is that it does not involve any disturbance of the temperature field by the introduction of a measuring appliance. The limit of application of this method is dependent on the intensity of the radiation and the sensitivity of the photographic plate: for the present stage of development of the method, the lower limit is about 250° C. and, of course, there is no upper limit.

Seger cones are small pyramids about 2½in. high, and are composed of a mixture of felspar, china clay, and quartz. They are so graded that they indicate a series of temperatures in steps of about 20° C. They are placed in suitable positions in the furnace where they can be seen through a sight-hole in the wall. As the temperature rises, each cone in turn gradually softens and bends over, the specified temperature of a cone being reached when the tip of the pyramid bends over sufficiently to touch the base of the support.

5.—Describe some form of resistance thermometer which is suitable for measuring temperatures in the neighbourhood of 1,000° C.

Answer.—The silicate and oxide of aluminium, as well as quartz and some carbides, i.e., ceramic substances, have the characteristic feature that they retain their chemical and physical stability up to the highest temperatures required for most industrial processes. They also have the peculiar characteristic of "semi-conductors," viz., the specific electrical resistance *decreases* as the temperature increases, and this change of resistance with temperature is surprisingly large so that they would appear to be well suited for use as resistance thermometers. The most suitable of all such semi-conductors appears to be sinter-corund which is a form of sintered alumina, exhibiting resistance variation

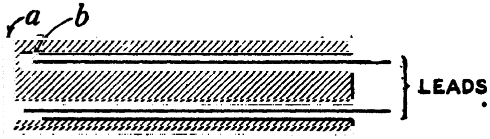


Fig. 9

with temperature, up to about 2,000° C., whereas quartz and porcelain, *for example, give a flattened characteristic at high temperatures.*

An important reason for selecting sinter-corund as a material for resistance thermometers is that it can be produced with a high degree of purity in a great variety of forms and retains its definite chemical characteristics in all the variety of its manufactured form. It is widely used, for example, for sparking-plugs owing to its high insulation quality and its great refractiveness. Up to 1,600° C. it retains a mechanical strength which is hardly reached by any other material, and it is extremely difficult to reduce or to alter its chemical stability in any other way. Its high heat-conductivity is an additional advantage for its use as a resistance thermometer because, for this reason, the time lag of the resistance change with temperature becomes correspondingly small.

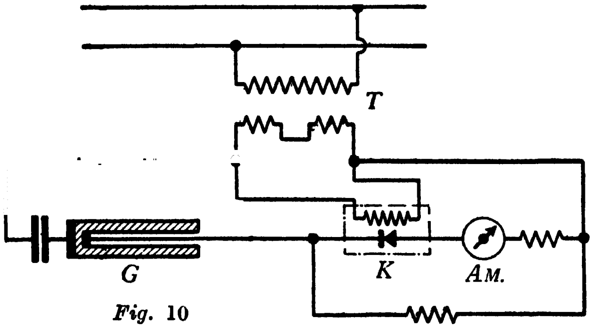


Fig. 10

The main feature of the construction of such a resistance thermometer will be seen by reference to Fig. 9. In order to eliminate the effects of changes in surface conductivity, the basic principle of the construction is so to arrange the electrodes that the path of the current is perpendicular to the surface. For this reason the thickness of the sinter-corund discs (*a* and *b* Fig. 9) utilised for the current path is very small in comparison with the adjacent walls which serve as insulation. Between the platinum electrodes and the sinter-corund is fused a layer of porcelain glaze about 0.5 mm. thick, thus ensuring that the electrodes become firmly bound to the sinter-corund dics. Tests made over periods of months have shown that within a temperature range of 800–1,100° C. the resistance characteristic remains perfectly constant.

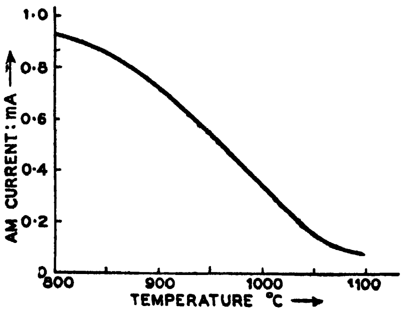


Fig. 11

The circuit connections for this type of resistance thermometer are shown in Fig. 10. A constant pressure supply of direct current is obtained from a single-phase transformer *T* by means of a vibrating reed rectifier *K* (see also Test Paper, Chapter X, Example 14). The moving coil ammeter *Am* gives a measure of the resistance of the sinter-corund thermometer *G*, from the calibration of which the corresponding temperature can then be read off.

Fig. 11 shows the relationship between the ammeter current readings and the temperature, from which it will be seen that, for a wide range of temperatures, this characteristic is very approximately a straight line.

6.—Give a diagram of connections for a Bridge method of measuring temperatures by means of a thermo-couple.

ANSWER.—The Bridge method shown in Fig. 12 is a null method and consequently is particularly well suited for the temperature control of annealing and hardening furnaces and drying ovens and also for recording the steam temperature, for example, in the inlet pipe of a turbine. The null method ensures that the measurement is independent of contact resistances and also of the resistance of the leads.

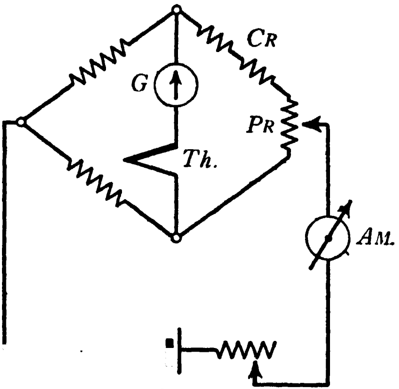


Fig. 12

In Fig. 12 the thermo-couple *Th* is shown in series with the galvanometer *G*. The resistance C_R is so constructed that it is dependent on the temperature, so that it can compensate for the temperature changes at the cold junction of the thermo-couple. The potentiometer resistance P_R enables the e.m.f. of the thermo-couple to be compensated so that no current will flow in the galvanometer *G*. If the galvanometer shows a deflection, the movement of its pointer is used to operate a mechanism which adjusts the potentiometer resistance P_R until balance has been restored, and the amount of such adjustment can be calibrated to provide a measure of the temperature of the hot junction of the thermocouple. The whole operation can be made automatic and incorporated with a temperature recording chart.

7.—What is the technical definition of a " black body " and what is its practical significance ?

ANSWER.—Heat waves are electromagnetic waves of the same nature as light waves from which they differ only by reason of their greater wavelength. Consequently, the well-known laws of optics are also applicable to the propagation, reflection, and refraction of heat waves. The following table gives, for comparative purposes, the more important ranges of electromagnetic waves:

Type of Radiation	Wavelength in microns μ
Short (gamma) waves	0.466×10^{-6}
X-rays	0.158×10^{-4} .... 660×10^{-4}
Ultra-violet rays	1.3×10^{-2} .... 36×10^{-2}
Visible light	0.36 .... 0.78
Ultra-red (heat) rays	0.78 340
Electric waves	340 .... ∞

$$1 \text{ micron } (\mu) = \begin{cases} 0.001 \text{ mm.} \\ 10^{-4} \text{ cm.} \\ 10^{-6} \text{ m.} \end{cases}$$

Also: $1\mu = 10^{4}$ Å (Ångström)

1 Å $= 10^{-8}$ cm.

The term " completely black " refers to those bodies which completely absorb all incident rays, and no such bodies exist. It is, however, possible to obtain substances which are practically " black " by means of a construction such as that shown in Fig. 13, which comprises a hollow space with a relatively small opening *S*. The inner walls of this hollow space are covered with a layer of material which is a powerful absorbent for heat rays so that such rays as may pass through the opening *S* will, after reflection at the inner walls be practically completely absorbed, that is to say, practically no heat rays will emerge from the opening *S*. The arrangement is used for radiation measurements.

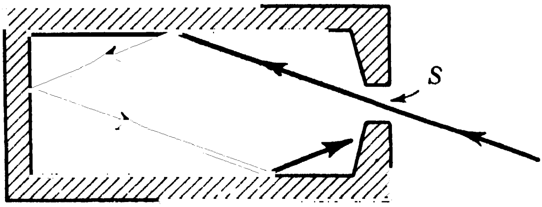


Fig. 13

In accordance with the foregoing, the conception " black " is not to be understood in the optical sense. Substances with bright surfaces can in certain circumstances be " black " in so far as heat radiation is concerned.

" Grey bodies " are those which reflect all wavelengths with equal strength.

" Coloured bodies " reflect selectively, that is, certain wavelengths will be reflected more strongly than others.

If, from an incident ray, only a single reflected ray is produced, the reflecting body is said to have a " smooth " surface; on the other hand, if a single ray is diffusely reflected, the surface is said to be " rough." The terms " smooth " and " rough " refer, of course, only to rays of definite wavelengths ; for example, a surface which is found to be " smooth " for long waves may be " rough " for very short waves.

Surfaces may be " rough " for very short waves because the unevenness of the surface in relation to the wavelength may not be negligible. If a " smooth " surface reflects all incident waves, it may be said to be a " mirror surface." If a " rough " surface reflects all incident waves, it is said to be a " white surface."

Chapter VII

MAGNETISM : MAGNETIC MEASUREMENTS

1.—Describe a method for measuring the horizontal component H of the earth's magnetic field.

ANSWER.—The value of the horizontal component H may be found in several ways, but only one method will be described—viz., that due to Gauss.

This method consists in measuring (1) the angle through which a very small, freely suspended, horizontal magnetic needle is deflected by a magnet in a given position, and (2) the period of vibration of the same magnet as used in (1), when suspended horizontally in the earth's field, and free to oscillate about a vertical axis.

The measurement of (1) gives an equation involving the ratio of the magnetic moment M of the magnet and the horizontal component H of the earth's field.

The measurement of (2) gives an equation involving the product of the same two quantities. Hence the value of H may be deduced from these two equations, and incidentally the same two equations give the value of the magnetic moment M of the magnet.

Experiment 1. *Determination of the Ratio* $\frac{M}{H}$.—In Fig. 1 let NS be the deflecting magnet of moment M. If this magnet is of a length $2l$, which is great compared with its lateral dimensions, say a cylindrical magnet of which the ratio $\frac{\text{length}}{\text{diameter}}$ is about 100, it is very approximately correct to assume that $2l$ is the distance between the poles, and that the strength of the poles is $\frac{M}{2l}$.

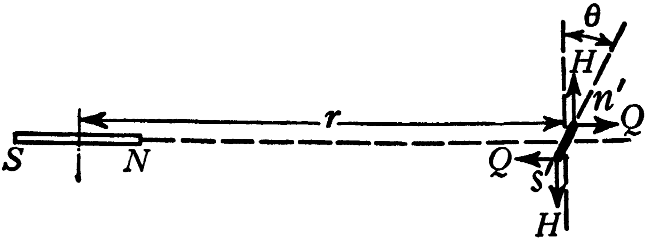


Fig. 1

Let $n's'$ be the small, freely suspended magnetic needle, the centre of $n's'$ being on the prolonged axis of NS, and let M' be the magnetic moment of the needle, and $2l'$ its length. If the ratio $\frac{\text{length}}{\text{diameter}}$ of the needle be sufficiently large, the strength of each pole may be taken as $\frac{M'}{2l'}$.

Let θ be the angle which the needle makes with the magnetic meridian when the relative positions of needle and magnet are as shown in Fig. 1.

The deflecting force on the pole n' will be the resultant of the repulsion due to N, and the attraction due to S. If the distance r between the centres of magnet and needle be great compared with l it will be very approximately correct to assume this resultant magnetic intensity Q at $n's'$ due to NS to be parallel to the line through SN, as shown in Fig. 1. The magnitude of the force on n' is given by the expression—

$$\frac{M'}{2l'}Q = \frac{MM'}{4ll'}\left[\frac{1}{(r-l)^2} - \frac{1}{(r+l)^2}\right] = M\frac{M'}{l'}\frac{r}{(r^2-l^2)^2},$$

that is
$$Q = M\frac{2r}{(r^2-l^2)^2}.$$

This will also be the resultant force on the pole s' of the attraction due to N, and the repulsion due to S. Hence the effect of the magnet NS on the needle $n's'$ is a deflecting couple of moment—

$$\frac{M'}{2l'}Q2l' \cos\theta.$$

This couple will be balanced by a couple due to the horizontal component H of the earth's field, which tends to set the needle in the magnetic meridian. The magnitude of this couple will be—

$$\frac{M'}{2l'}H2l' \sin\theta = M'H\sin\theta.$$

For equilibrium of the needle these two couples must be equal.

Hence
$$\frac{M'}{2l'}Q2l^1 \cos\theta = M'H\sin\theta;$$

that is
$$\tan\theta = \frac{QM'}{M'H} = \frac{Q}{H}$$

or
$$\tan\theta = \frac{M}{H}\frac{2r}{(r^2-l^2)^2};$$

and
$$\frac{M}{H} = \frac{(r^2-l^2)^2}{2r}\tan\theta.$$

If r is very large compared with l this expression reduces to—

$$\frac{M}{H} = \frac{r^3}{2} \tan \theta.$$

It is advisable to take another reading for θ by turning the magnet NS end for end, and the mean of the two readings is then taken as the true value of the deflection.

The values of θ may be measured by the movement of a spot of light reflected on to a scale from a mirror attached to the needle $n's'$.

Experiment 2. *Determination of the Product MH.*—For this purpose the magnet which was used in Experiment 1 for the purpose of deflecting the small suspended needle is now hung in a horizontal position by means of a single fibre of unspun silk, precautions being taken that there is no torsion on the fibre when the magnet is at rest in its position of equilibrium. The magnet is then deflected by approaching a small magnet, care being taken that a true oscillation is obtained, and not combined with a pendulum motion. The time of a complete oscillation is taken when the amplitude of vibration has died down to a small amount, say about 3° deflection each side of the neutral position.

If θ represents the angular deflection of the oscillating magnet from the neutral position, then for small values of θ and neglecting the damping effect of the air, etc., the equation of motion is—

$$\frac{d^2\theta}{dt^2} = -\frac{MH}{K}\theta,$$

where K is the moment of inertia of the magnet round an axis through its centre and at right angles to its length.

The solution of this equation is—

$$\theta = A \sin\left[\sqrt{\frac{MH}{K}}\, t - \alpha\right],$$

where A and α are constants.

The time of a complete oscillation is therefore—

$$T = 2\pi\sqrt{\frac{K}{MH}}.$$

Since, as was assumed in Experiment 1, the diameter of the magnet is small compared with its length the moment of inertia may be written—

$$K = \frac{Wl^2}{3},$$

where W is the mass of the needle.

Hence $$MH = \frac{4\pi^2 l^2 W}{3T^2}.$$

Combining this equation for the product MH with the equation given by Experiment 1 for the quotient $\frac{M}{H}$ gives—

$$H^2 = \frac{8}{3}\,\frac{\pi^2 l^2 r W}{T^2(r^2 - l^2)^2 \tan\,\theta};$$

or, neglecting l compared with r—

$$H^2 = \frac{8}{3}\,\frac{\pi^2 l^2 W}{r^3 T^2 \tan\,\theta}.$$

Similarly $$M^2 = \frac{2}{3}\,\frac{\pi^2 r^3 l^2 W \tan\,\theta}{T^2}.$$

The data of the earth's field in England are—

H = 0·18 : dip $\delta = 67°\ 14'$: intensity F = 0·475.

For a more detailed account of this and other standard measurements the reader is referred to Professor A. Gray's book on " Absolute Measurements in Electricity and Magnetism."

The total intensity F of the earth's field is connected with the dip δ and the horizontal component H by the expression—

$$H = F \cos\,\delta.$$

See also example 6 Test paper for Chapter VIII.

2.—*Find the potential at a point P in the field of a small bar magnet.*

Answer.—Since Coulomb's Laws for the force between two magnetic quantities and the force between two electrical quantities respectively, are identical in form, many of the fundamental propositions which apply to electric fields of force also apply to magnetic fields of force. Thus, since the potential at a point distant r cms. from an electric charge q is $\frac{q}{\varepsilon r}$ units, the potential at a point distant r cms. from a magnetic quantity m is $\frac{m}{\mu\, r}$ units, where, for air, the dielectric constant ε and the magnetic permeability μ are each unity.

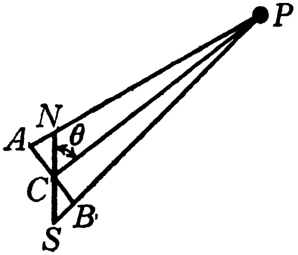


Fig. 2

Referring now to Fig. 2 in which $N\ S$ is a small bar magnet of length l cms. and pole strength m units, the distance between some given point P and the centre C of the magnet being r cms. The potential at P due to the N pole of the magnet is $\frac{m}{NP}$ and due to the S pole of the magnet is $\frac{m}{SP}$

so that the resultant potential at P due to both poles is,

$$v = \frac{m}{NP} - \frac{m}{SP} \text{ c.g.s. units.}$$

(See page 87, expression (14) of " Principles of Electrical Engineering.")

Now draw ACB at right angles to CP and draw PNA and PBS. If then CP is large as compared with NS the lines NA and SB will be very approximately at right angles to AB; also, PA, PB, and PC will be very approximately equal—that is, $PA = PB = r$ very approximately. Hence,

$$PN = PA - AN = r - \frac{l}{2}\cos\theta$$

$$PS = PB + BS = r + \frac{l}{2}\cos\theta$$

so that the potential at P is—

$$v = m\left\{\frac{1}{r - \frac{l}{2}\cos\theta} - \frac{1}{r + \frac{l}{2}\cos\theta}\right\}$$

or

$$v = \frac{m\,l\cos\theta}{r^2 - \left(\frac{l}{2}\right)^2\cos^2\theta} = \frac{M\cos\theta}{r^2}$$

Where $M = ml$ and is the moment of the magnet NS and $\left(\frac{l}{2}\right)^2$ is negligibly small compared with r^2.

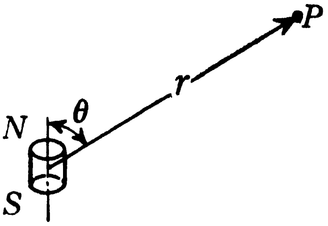


Fig. 3

If the magnet is in the form of a small cylindrical bar having its magnetism concentrated on its end faces as shown in Fig. 3, the area of each end being δS, the magnetic intensity being $\mathcal{J}$, and the length of the bar being l cms, then the potential at P will be

$$v = \frac{M\cos\theta}{r^2} = \frac{\mathcal{J} \times l \times \delta S\cos\theta}{r^2}.$$

Now since l is very small the solid angle subtended at P by either end face δS is

$$\delta\omega = \frac{\delta S\cos\theta}{r^2}$$

so that the potential at P is

$$v = \mathcal{J}l\,\delta\omega \text{ c.g.s. units. } \dots\dots\dots\dots\dots\dots \quad (1)$$

If the small cylindrical magnet of Fig. 3 is regarded as an element of a " magnetic shell " (see example 3) this result can be expressed by stating that the potential at P is given by the strength of the shell multiplied by the solid angle subtended at P by this element of the magnetic shell.

3.—*What is a " magnetic shell of uniform strength," and what is its practical significance ?*

Answer.—Imagine a thin sheet of magnetisable material of uniform thickness and uniformly magnetised in a direction perpendicular to its surface, such a magnetised sheet is a " magnetic shell of uniform strength." The importance of the investigation of the properties of a magnetic shell lies in the fact that such a shell is equivalent in its magnetic effects to a current of a certain strength flowing in an electric circuit of which the shape is the same as the boundary edge of the shell.

This equivalence of a uniform magnetic shell and electric circuit was first stated by Ampère and all subsequent observations and experiments have confirmed the statement. The results deduced for a magnetic shell, e.g., potential at a point outside the shell, force on the shell when placed in a magnetic field etc., can be immediately applied to the equivalent electric circuit.

The properties of the magnetic shell can be deduced from Coulomb's Law and although such a shell does not actually appear in practice, the fact that it is equivalent to an electric current circuit means that the investigation of the properties of a magnetic shell is also an investigation of the magnetic properties of an electric circuit and in this way, a linking up of the phenomena of magnetism and electric currents is obtained.

The strength of a magnetic shell is the product of the intensity of magnetisation and the thickness of the shell, or, otherwise stated, is the magnetic moment per unit area. If the thickness is t cm. and the intensity of magnetisation is J—this being also the density of magnetism on the surface of the shell—then the strength of the shell is

$$\Phi = Jt$$

4.—*Find the potential of a magnetic shell at a point P.*

Answer.—In Fig. 4 consider a filament AB of the shell taken in the direction of magnetisation—that is perpendicular to the surface of the shell. Let δS sq. cms. be the element of surface on each side of the shell forming the ends of the filament and let P be distant r cms. from the centre C of the filament. If A is magnetised with an intensity $+J$ and B magnetised with an intensity $-J$ then the magnetism

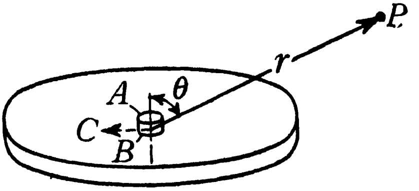


Fig. 4

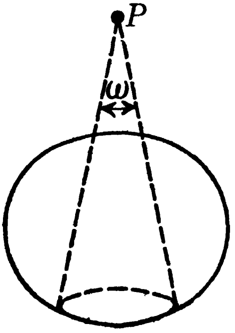


Fig. 5

on the end surface A of the filament is $J\delta S$ and the end surface B is $+J\delta S$. Three cases will be considered, viz.: (i) the point P is on the positively magnetised side of the shell; (ii) P is on the negatively magnestised side; (iii) P is within the material of the shell:

Case (i): The point P is on the positively magnetised side.—This condition is shown in Fig. 5, and the potential at P due to an elementary magnet such as AB in Fig. 4 is:

$$v = \frac{Jt\ \delta S \cos\theta}{r^2} = \Phi \frac{\delta S \cos\theta}{r^2}.$$

Where θ is the angle which CP makes with the axis of the magnet (Fig. 4) the end A being positively magnetised as already stated.

If $\delta\omega$ is the solid angle which the end surface A subtends at P then $\delta\omega = \frac{\delta S \cos\theta}{r^2}$ and the potential at P is $\Phi\,\delta\omega$, that is, the strength of the shell multiplied by the solid angle subtended at P by the element of the shell (see also Example 2). A similar result holds for each element of the shell so that the potential at P due to the whole shell is equal to the strength of the shell multiplied by the solid angle subtended at P by the whole shell, i.e. $\Phi\omega$.

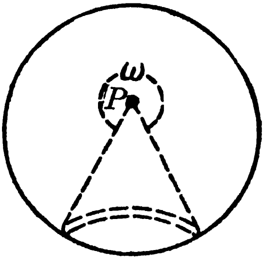


Fig. 6

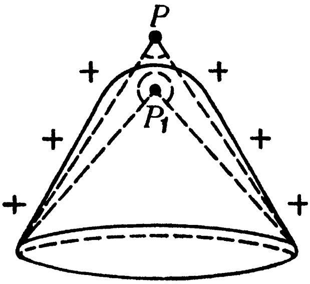


Fig. 7

The solid angle subtended at P by the shell is the solid angle subtended by the boundary of the shell so that the potential at P depends only on the strength of the shell and the shape and position of the boundary of the shell. If the shell is completely closed then $\omega = 0$, so that the potential at a point outside a completely closed shell is zero.

Case (ii): The point P is on the negatively magnetised side of the shell, as shown in Fig. 6.—By exactly similar considerations as those of case (i) it will be seen that the potential at the point P on the negatively magnetised side is $-\Phi\,\omega$. If the shell is completely closed and the point P lies inside, the solid angle will be 4π, and hence the potential at a point inside a closed shell is $\mp 4\pi\,\Phi$ according as to whether the inside surface is negatively or positively magnetised.

It is also to be noted that if, in Fig. 7, P and P, be points close together,

P being on the positively magnetised side and P_1 on the negatively magnetised side, then

$$\omega = 4\pi - \omega_1$$

where ω is the solid angle subtended at the boundary by P and ω_1 by P_1.

Case (iii): The point P lies inside the material of the shell.

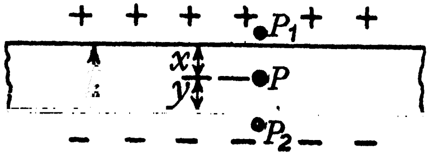


Fig. 8

Suppose P (see Fig. 8) is distant x from the positively magnetised side and y from the negatively magnetised side, then P may be considered as being on the negatively magnetised side of a shell of thickness x and on the positively magnetised side of a shell of thickness y, from which it follows that the potential at P is

$$v = Jy\omega - (4\pi - \omega)Jx = J\omega t - 4\pi Jx.$$

or

$$v = \Phi\omega - 4\pi Jx.$$

If $x = 0$: P comes to P_1 and $v = \Phi\omega$ as in Case (i).

If $x = t$: P comes to P_2 and $v = -(4\pi - \omega)\,\Phi$ as in Case (ii)

so that in passing from P_1 to P_2 the potential changes by the amount

$$\omega\Phi + 4\pi\Phi - \omega\Phi = 4\pi\Phi.$$

These results should be compared with those which are derived on page 215 of "Principles of Electrical Engineering."

5.—*What change of direction takes place when lines of magnetic induction pass from one medium into another?*

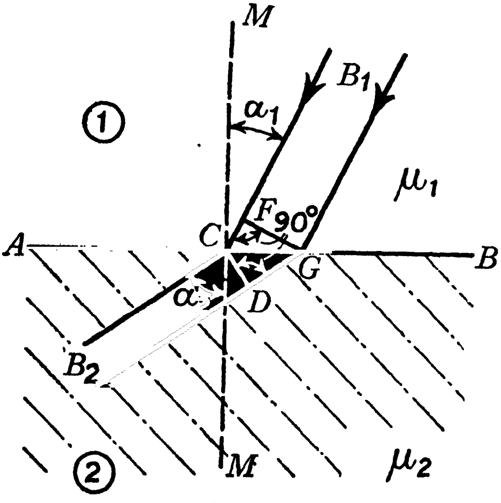


Fig. 9

ANSWER.—In Fig. 9 is shown a tube of magnetic induction B_1 passing from the medium ① to medium ② across the boundary AB, the flux in this tube remaining constant as it crosses the boundary.

Then:

Magnetic Flux

$$= CD \times B_2 = FG \times B_1$$

that is,

$$B_2\, CG \times \cos\alpha_2 = B_1\, CG \times \cos\alpha_1$$

or

$$\frac{B_1}{B_2} = \frac{\cos\alpha_2}{\cos\alpha_1}$$

Further, if, for the closed path shown in Fig. 10, the points *a*, *b*, *c*, *d* are very close to the boundary surface, the potential difference between *a* and *c* is indefinitely small, as is also the p.d. between *b* and *d*. Hence, since the work done in moving unit magnetic pole round the closed path *a*, *b*, *d*, *c* is zero, the p.d. between *a* and *b* must be equal to the p.d. between *c* and *d* and consequently,

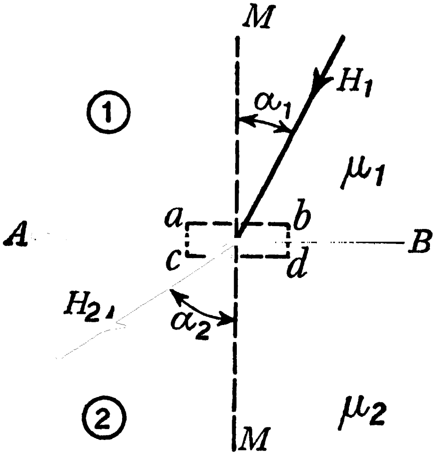


Fig. 10

$$H_1 \sin \alpha_1 = H_2 \sin \alpha_2.$$

Combining this relationship for the magnetic force with the one already obtained for the magnetic induction it is seen that

$$\frac{B_1 \, Cos \, \alpha_1}{H_1 \sin \alpha_1} = \frac{B_2 \, Cos \, \alpha_2}{H_2 \sin \alpha_2}$$

that is $$\mu_1 \text{ Cot } \alpha_1 = \mu_2 \text{ Cot } \alpha_2$$

or $$\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\mu_2}{\mu_1}$$

A corollary of this result is that magnetic lines of force in air will enter or leave a highly magnetic material (*e.g.* a pole shoe) very nearly at right-angles to the surface.

Suppose, for example, that the medium ① Fig. 10 is air so that $\mu_1 = 1$, then

$$\frac{\tan \alpha_2}{\tan \alpha_1} = \mu_2$$

If the medium ② Fig. 10 has a permeability $\mu_2 = 300$ the relationship

between the angles α_2 and α_1 is given in the Table I. In this Table are also given the corresponding value of the angles α_1 and α_2 for $\mu_2 = 3000$.

TABLE I

$\mu_2 = 3000$		$\mu_2 = 300$	
α_2 (iron)	α_1 (air)	α_2 (iron)	α_1 (air)
0°	0°	0°	0°
10°	0° 0′ 1″	10°	0° 2′
30°	0° 0′ 40″	30°	0° 7′
50°	0° 1′ 30″	50°	0 14′
70°	0° 3′	70°	0° 32′
80°	0° 6′	80°	1° 5′
85°	0° 13′	85°	2° 11′
89°	1° 5′	89°	10° 45′
89½°	6° 36′	89½°	48° 45′
90°	90°	90°	90°

6.—Explain the principle in accordance with which a space may be screened from a magnetic field.

ANSWER.—It has been seen in the solution to the Example (5) how the lines of magnetic induction become deflected when they cross the boundary between two media at any angle which is not 90° to the boundary surface, and the Table there given shows how the amount of this deflection depends upon the permeabilities of the respective media.

A little consideration will show that a consequence of this effect is that if an iron tube is placed in a uniform magnetic field in air, the lines of magnetic induction will be deflected as shown in Fig. 11, so that the space within the tube is magnetically screened. The extent to which this screening can be realised in practice depends upon the radial thickness of the iron wall of the tube and on the permeability of this iron wall.

If a spherical iron shell of permeability μ is placed in a uniform magnetic field in air (see Fig. 12), it can be proved that the intensity of the magnetic force H_0 within the hollow space is given by the expression—

$$H_0 = \frac{H}{1 + \frac{2}{9}\frac{(\mu - 1)^2}{\mu}\left[1 - \beta\right]}$$

where, $\beta = \left(\frac{a_1}{a_2}\right)^3$

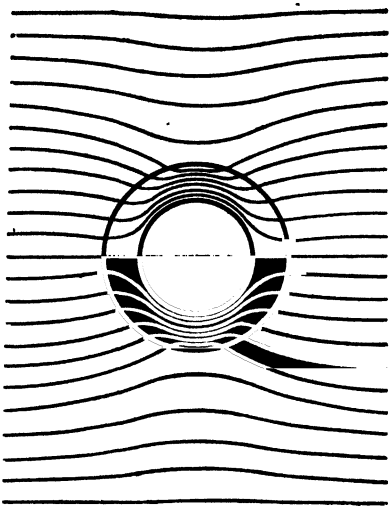


Fig. 11

thus, if $\mu = 3000 : a_2 = 2a_1$

$$H_0 = \frac{H}{1 + \frac{6{,}000}{9}\left[1 - \frac{1}{8}\right]} = \frac{H}{590}$$

An iron shield surrounding a galvanometer is sometimes used to protect the galvanometer from external magnetic fields. The magnetic shielding of cathode-ray tubes is also effected by means of a tube of high permeability material such as mu-metal.

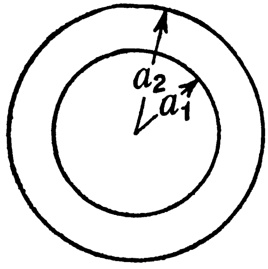


Fig. 12

7.—*Discuss the magnetic force conditions which will exist when two magnetised surfaces are placed parallel to each other in a gaseous or liquid material of permeability* μ_1.

ANSWER.—If a magnetised surface *AB*, Fig. 13, is placed so that it is parallel to the surface *CD* of a solid magnetisable material of permeability μ_2 the medium between the two surfaces *AB* and *CD* being a gaseous or liquid material of permeability μ_1, there will then be a force either of attraction or repulsion acting between the two surfaces *AB* and *CD*. The direction of this force can be found from the following considerations.

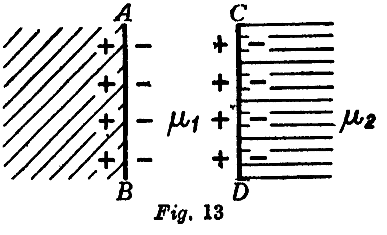


Fig. 13

The energy stored in the gaseous or liquid medium of permeability μ_1, that is, in the space between the two surfaces AB and CD will be,

$$\frac{B^2}{8\pi\mu_1} \text{ ergs per c.cm.}$$

and the energy stored in the solid magnetisable material of permeability μ_2 will be

$$\frac{B^2}{8\pi\mu_2} \text{ ergs per c.cm.}$$

Now the direction of the force between the two surfaces AB and CD will be such as to tend to make the total energy of the system a minimum. If the permeabilities of the respective materials are such that $\mu_1 < \mu_2$ then the energy of the system will be reduced if the volume of the space between the two surfaces AB and CD is reduced: it follows from this that the force between the two surfaces will be one of attraction. Conversely, if $\mu_1 > \mu_2$ the energy of the system will be reduced if the space between the two surfaces is increased, so that in this case the force will be one of repulsion.

8.—What mechanical force is necessary to drag a block of iron from the end of a magnetised bar of circular section and 2 cm. diameter, when the induction density at the end of the bar is B = 3,200 gauss ?

ANSWER.—The energy stored in a magnetisable material is

$$U = \tfrac{1}{2}\,\frac{BH}{4\pi 10^7} \text{ joules per c.cm.}$$

If P is the required mechanical force and S sq. cm. the area of the end of the bar, and if $\triangle$cm. is the thickness of the air layer between the two surfaces, viz., the end surface of the bar and the surface of the block, then

$$P \times \triangle = \tfrac{1}{2}\,\frac{BH}{4\pi 10^7}\, S \triangle \text{ joules}$$

or

$$P = \left(\tfrac{1}{2}\,\frac{BH}{4\pi 10^7}\, S\right) \text{ joules/cm.}$$

If P is in kg. then, since 1 kg.m. = 9.81 joules, 1 kg. cm. $= \frac{9{\cdot}81}{10^2}$ joules,

so that

$$P \text{ (in kg.)} = \left(\tfrac{1}{2}\,\frac{BHS}{4\pi 10^7}\right)\frac{10^2}{9{\cdot}81}$$

inserting $B = 3{,}200$: $H = B$, since $\mu = 1$, for air

$$S = \frac{\pi}{4}2^2 = \pi \text{ sq. cm.}$$

gives

$$P = 1{\cdot}3 \text{ kg.}$$

9.—Successive deflections of a ballistic galvanometer with small damping are as shown in Fig. 14, viz.: A_1: A_2: A_3 . . . , where

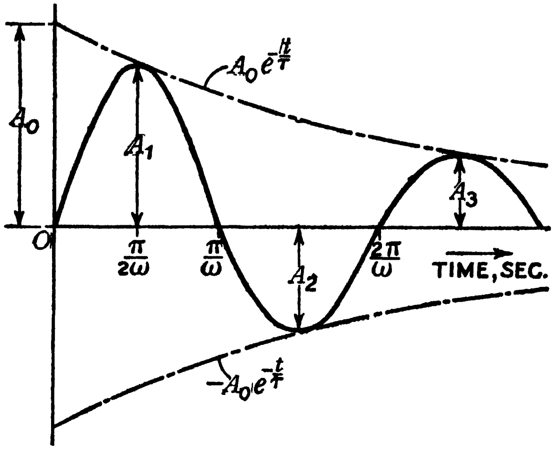


Fig. 14

$A_1 = 10$ cm.: $A_3 = 9{\cdot}5$ cm., and so on. Find the undamped deflection A_0 from the following data:

(i) The logarithmic decrement is λ nepers per ½ period of swing.

(ii) The logarithmic decrement is $\triangle$ nepers per period of swing.

Answer.—The general equation for such a lightly damped oscillation is

so that $\quad A_0\, e^{-\frac{t}{\tau}} \sin \omega t$:

and the periodic time of oscillation is $T = \dfrac{2\pi}{\omega}$ sec.

$$A_1 = A_0\, e^{-\frac{1}{\tau}\frac{\pi}{2\omega}} : A_2 = A_0\, e^{-\frac{1}{\tau}\frac{3\pi}{2\omega}} : A_3 = A_0\, e^{-\frac{1}{\tau}\frac{5\omega}{2\pi}}$$

and $$\frac{A_1}{A_2} = e^{\frac{1}{\tau}\frac{\pi}{\omega}} : \frac{A_1}{A_3} = e^{\frac{1}{\tau}\frac{2\pi}{\omega}}$$

(*i*) A_0 derived from the log. dec. λ nepers per ½ period of swing.

$\lambda = \log_e \dfrac{A_1}{A_2} = \frac{1}{2} \log_e \dfrac{A_1}{A_3} = \left(\dfrac{1}{n-1}\right) \log_e \dfrac{A_1}{A_n}$ where A_n is the *nth* peak value. That is

$$\lambda = \log e^{\frac{1}{\tau}\frac{\pi}{\omega}} = \frac{1}{\tau}\frac{\pi}{\omega} = \tfrac{1}{2}\log_e \frac{10}{9{\cdot}5} = \tfrac{1}{2} \times 2{\cdot}3 \log_{10} \frac{10}{9{\cdot}5}$$

or

$$\underline{\lambda} = 1{\cdot}15 \log_{10} \frac{10}{9{\cdot}5} = \underline{0{\cdot}0253}$$

so that

$$A_0 = A_1 e^{\frac{1}{2}\frac{1}{\tau}\frac{\pi}{\omega}} = A_1 e^{\frac{1}{2}\lambda} \simeq A_1 (1 + \tfrac{1}{2}\lambda) = 10 \times 1{\cdot}0126$$

or

$$\underline{A_0 = 10{\cdot}126}$$

(*ii*) A_0 derived from the log. dec. $\triangle$ nepers per period of swing.

$$A_1 = A_0 e^{-\frac{1}{\tau}\frac{\pi}{2\omega}} : A_3 = A_0 e^{-\frac{1}{\tau}\frac{5\pi}{2\omega}}$$

$$\frac{A_1}{A_3} = e^{\frac{1}{\tau}\frac{2\pi}{\omega}} :$$

$$\triangle = \log_e \frac{A_1}{A_3} = 2\frac{1}{\tau}\frac{\pi}{\omega} = 2.3 \log_{10} \frac{10}{9.5}$$

so that

$$\underline{\triangle = 0.0505 = 2\lambda}$$

and

$$A_0 = A_1 e^{\frac{1}{\tau}\frac{\pi}{2\omega}} = A_1 e^{\frac{1}{4}\triangle} \simeq A_1 (1 + \tfrac{1}{4}\triangle)$$

hence,

$$\underline{A_0} = 10\,(1{\cdot}0126) = \underline{10{\cdot}126}$$

as in (*i*).

10.—*Explain the principle of action of the thermo-magnetic motor.*

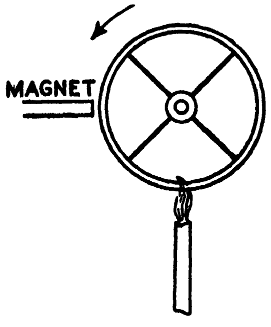


Fig. 15

ANSWER.—In Fig. 15 is shown a wheel, the rim of which is of sheet nickel, the spindle being mounted in a horizontal plane so that the wheel is free to turn. A flame is held against the lower part of the rim and heats it, in consequence of which the magnetic permeability of the lower part of the rim is reduced relatively to that of the upper portion of the rim. If now a magnet is held as shown in Fig. 15, the magnetic attraction which will act on the cool upper portion of the rim will be greater than that on the heated lower portion, so that the wheel is maintained in rotation in the direction shown by the arrow. This is an example of the conversion of heat into mechanical energy without the use of some intermediate fluid such as steam or gas. The disadvantage of the thermo-magnetic motor is that its efficiency of energy conversion is extremely small.

11.—Show how the principle of magneto striction may be applied to the measurement of Young's modulus for a rod or wire of iron, nickel, or cobalt.

ANSWER.—The wire under test is supported symmetrically in a long solenoid which is excited by direct current. Arranged at the central part of the solenoid and embracing the wire is a search coil of a large number of turns, e.g., 50,000. The search coil is connected through a valve amplifier to an oscillograph, the inclusion of the valve amplifier making it possible to operate the oscillograph without taking any appreciable current from the search coil.

The wire is set in a state of longitudinal vibration either by drawing a piece of resin-coated leather along its surface or, alternatively, by means of a gentle tap applied to one end. When the solenoid is excited and the wire is set in a state of longitudinal vibration, the consequent extension and compression of the wire produces corresponding changes in the permeability, so that the flux which is linked with the search coil changes, and an e.m.f. is developed which is of the same frequency as that of the vibrations of the wire. If an oscillographic record is taken of the e.m.f. induced in the search coil and if, simultaneously, a photographic record is taken of a wave of known frequency, a time scale is obtained and the frequency of the induced e.m.f. in the search coil and consequently, the frequency of the vibrations of the wire, can be determined. In Fig. 16 is shown an example of such an oscillogram, and the main relevant data are also given on page 106.

In order to derive the value of Young's modulus from such a record it is to be noted that the natural frequency of the longitudinal vibrations of a wire which is supported at one end is given by the expression,

$$f = \frac{c}{4l} \text{ vibrations per second,}$$

where c is the velocity of sound in the wire in cm. per sec. and l cm. is the free length of the wire. Since the velocity of sound in the wire is related to Young's modulus E and the density ρ of the wire by the expression $c^2\rho = E$, the formula for the natural frequency of vibration may be written

$$f = \frac{1}{4l}\sqrt{\frac{E}{\rho}} \text{ vibrations per second.}$$

where E is expressed in dynes per sq. cm. and ρ is the density in gm. per c.cm. It follows, therefore, that

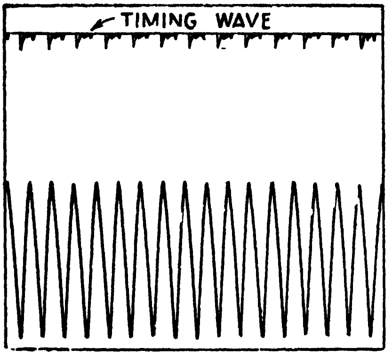


Fig. 16

$E = \rho(4fl)^2$ dynes per sq. cm.

or

$E = 14{\cdot}5 \times 10^{-6} \times \rho(4fl)^2$ lb. per sq. in.

For the test to which the oscillogram of Fig. 16 refers, the frequency of the $\frac{1}{4}''$ diameter Armco iron rod was found to be 646 cycles per second, the density of the iron was 7·85, and the freely vibrating length was 199·5 cms., then

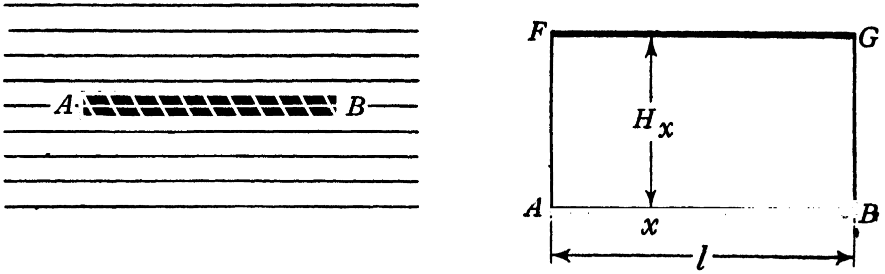
$$E=\rho(4fl)^2=20{\cdot}9\times10^{11} \text{ dynes per sq. cm.}$$
$$=30{\cdot}3 \times 10^6 \text{ lbs. per sq. inch.}$$

For further information on this subject see Journal of the I.E.E., Vol. 75, December, 1934, page 784 (T. F. Wall).

12.—*Explain the principle of operation of the magnetic-potential meter.*

Answer.—Fig. 17*a* shows a uniform magnetic field of intensity H and placed in this field with its axis parallel to the direction of H oersted is the flexible magnetic-potential coil AB of length l cms. Under these conditions, each turn of this coil will embrace the same magnetic field and the magnetic potential between the ends A and B will then be :

$$H.l \text{ ergs.}$$



(a) (b)

Fig. 17

Referring now to Fig. 17*b*, in which AB represents the axis of the coil, and the ordinate H_x, at any point x, represents the magnetic potential at that point. The magnetic flux which threads each turn of the coil will be

$$\Phi = H.Q \text{ maxwell,}$$

when Q sq. cm. is the mean area of each turn of the coil. The total *flux-linkages* of the coil will be given by the expression

$$J = \Phi \, n.l. = H.Q.n.l. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Where n is the number of turns per cm. length of the coil, this quantity Φ can be measured directly by means for example, of a ballistic galvanometer. The magnetic potential difference between the two ends A and B of the coil in Fig. 17*a* will then be

$$H.l = \frac{\text{Flux-linkages } J}{Q.n.} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Next suppose that the flexible coil AB remains straight as indicated by the horizontal line AB in Fig. 18, but that the component of magnetic intensity in the direction of the axis of the coil is no longer uniform as in Fig. 17*a*, but is now represented at each point along the coil by the ordinates of the curve FKG. If the magnetic potential coil has only a few turns, say, for example, 7 equi-spaced turns, then if at the point f, which is midway between the two consecutive turns 1 and 2, the ordinate fg is drawn, then the mean value of the flux which is embraced by each of the coils 1 and 2 will be given by the shaded area of the rectangle, as shown in Fig. 18. Similarly, if corresponding ordinates are drawn, each midway between two neighbouring turns, the stepped rectangular diagram of Fig. 18 will be obtained, and the area of this rectangular diagram will be a first approximation to the area of the diagram $AFKGB$. If the number of equi-spaced coils is increased, the stepped rectangular figure will become a closer approximation to the area $AFKGB$, and when the number of equi-spaced coils becomes indefinitely large, the stepped diagram of Fig. 18 will coincide with the smooth curve FKG. That is to say, the mean ordinate of the magnetic intensity curve FKG will be (see expression (1)),

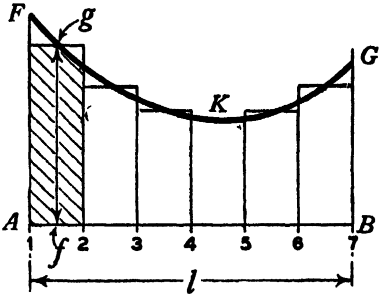


Fig 18

$$H_{\text{av}} = \frac{\Sigma\,(\Phi n)}{Q.n.l} = \frac{J}{Q.n.l} \text{ oersted.}$$

Where $J = \Sigma\,(\Phi.n)$ and is the total number of flux-linkages of the flexible coil. Consequently, the magnetic potential between the two ends A and B of the coil will be

$$H_{\text{av}}.\, l = \frac{J}{Q.n} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Similar considerations show that if the flexible coil is bent, as shown for example in Fig. 31, page 205 of " Principles of Electrical Engineering " the difference of magnetic potential between the two ends of the coil will be

$$H_{av}.\, d = \frac{\text{Flux-linkages}}{Q.n} \quad \dots\dots\dots\dots\dots\dots \quad (4)$$

where d cm. is the distance between the two ends of the coil. If H is the intensity of the magnetic force in the iron between the two ends of the coil, then $H = H_{av}$ as given by the expression (4).

If the calibration of the flexible coil is carried out by the method described on page 204 with reference to Fig. 31 of " Principles of Electrical Engineering " so that when the number of ampere-turns of the magnetising coil is $(AW)_{cal}$. the throw on the scale of the ballistic galvanometer is $\alpha°$, then a throw of $X°$ will correspond to a magnetic potential difference of

$$\frac{X}{\alpha} \frac{4\pi}{10} (AW)_{cal}$$

so that the required value of H will then be given by

$$H.d = \left(\frac{X}{\alpha} \frac{4\pi}{10} (AW)_{cal} \right) \frac{1}{d} \text{ oersted,} \quad \dots\dots\dots\dots \quad (5)$$

in which expression, d cm. is the distance between the two ends of the flexible coil as has been stated already in the foregoing.

Although the principle of the magnetic-potential meter has been known for about thirty years, it is only comparatively recently that its practical significance for testing the magnetic qualities of iron and iron alloys has begun to be realised. Exhaustive tests have shown, for example, that it is possible to obtain results of a remarkably high degree of accuracy in tests of the magnetic properties of single crystals of iron.*

13.—*On page 204, Fig. 30 of " Principles of Electrical Engineering " the method is explained for calibrating a magnetic-potential meter flexible coil. Describe two methods for calibrating the non-flexible semi-circular coil system shown in "Principles of Electrical Engineering" on page 205, Fig. 31.*

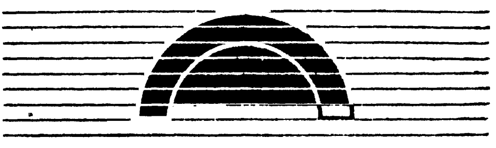


Fig. 19

ANSWER—*Method* 1.—The coil system is arranged in a magnetic field of uniform intensity as shown in Fig. 19 ; for example, at the middle of a

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(* See " Archiv für Elektrotechnik," 1929, page 145.)

long, straight solenoid. If there are w turns per cm. length of the solenoid winding and the solenoid current is I amperes, then the intensity of the magnetic field at and near the middle of the solenoid will be

$$H = \left(\frac{4\,\pi}{10} I\,w \right) \text{oersted.}$$

If, then, L cms. is the mean span of the magnetic-potential coil system the magnetic potential difference between the ends of the coil will be

$$H.L. = \frac{4\pi}{10} I.\omega.L \text{ oersted-cm.}$$

If α mm. is the throw on the ballistic galvanometer scale when the solenoid current is reversed, then a throw of 1 mm. represents a potential difference between the ends of the coil of

$$\frac{1}{\alpha} \left(\frac{4\pi}{10} I.w.L \right) \text{oersted-cm.}$$

If, then, the magnetic potential coil is in use as shown in Fig. 32, page 205 of " Principles of Electrical Engineering," and if when the magnetising coil current is reversed the throw on the ballistic galvanometer scale is X mm., the magnetic potential difference between the ends of the coil will be

$$H.L = \frac{X}{\alpha} \frac{4\pi\ I.w.L}{10}$$

and consequently the magnetic intensity which is being measured is

$$H = \frac{\text{X}}{\alpha} \frac{4\pi.I.w}{10} \text{ oersted.} \quad \ldots.(1)$$

A great advantage of this method of calibration is that the span L of the coil is taken into account in the calibration constant.

Method 2.—For this method the coil is arranged in the magnetic field due to a current I amperes in a straight wire. The magnetic-potential coil is arranged to stand on a flat horizontal surface, the plane of which passes through the axis of the wire as shown in Fig. 20.

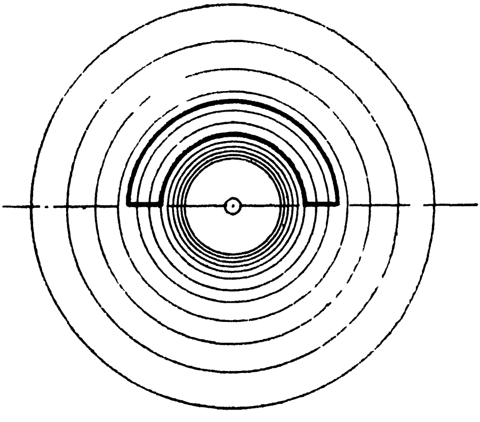

Fig. 20

If H oersted is the magnetic intensity in the mean semi-circular path through the coil, and if s cm. is the length of this path, then the magnetic potential difference between the ends of the coil will be

$$H.s = \frac{1}{2}\left(\frac{4\pi}{10} I\right) \text{ oersted-cm.} \qquad (1)$$

If α mm. is the throw of the galvanometer scale when the current I is reversed, then a throw of 1 mm. represents a magnetic potential difference between the ends of the coil of

$$\frac{1}{2\alpha}\left(\frac{4\pi}{10} I\right) \text{ oersted-cm.} \qquad (2)$$

This method of calibration has the disadvantage that a large current I must be used, and the section of the wire must be correspondingly large. The arrangement may therefore be modified by using a number of individual wires laid side-by-side on the plane through the centre of the magnetic-potential coil and connecting these wires in series. Such an arrangement is permissible because perfect symmetry with respect to the coil of the field due to any one wire is not necessary ; what is essential, however, is perfect symmetry of the field with respect to the plane on which the coil stands, and this is obtained by the arrangement described.

If, then, there are N such wires in series, laid side-by-side, and each carrying a current of I amperes, the required calibration is given by the following modification of the expression (2), viz.,

A throw of 1 mm. on the scale of the ballistic galvanometer will represent a magnetic potential difference between the two ends of the coil of

$$\frac{1}{2\alpha}\left(\frac{4\pi}{10} I.N\right) \text{ oersted-cm.} \qquad (3)$$

W. Wolman (Arch. f. Elek., 1928, page 385), has carried out a large number of magnetic investigations with a rigid semi-circular coil system of which the leading particulars are as follows :

The coil winding consists of 10 layers of 0.1 mm. diameter, enamel insulated, copper wire, the essential data of the coil system being :

Span of coil .. .. = 30 mm.
Breadth of coil .. .. .. = 16 mm.
Mean cross-section of the coil .. = 0.6 cm^2.
Number of turns per cm. length of coil = 700
Accuracy .. .. .. ≏ = 0.5 per cent.
Sensitivity .. .. .. H = 0.146 oersted per mm. throw with a scale distance of 1 metre.

14.—Describe the procedure for measuring static magnetic fields by means of a "bismuth spiral."

Answer.—For the measurement of stationary magnetic fields two main methods are available in practice. One of these makes use of the ballistic impulse currents which are induced in a test (i.e., search) coil when the magnetic field is suddenly reversed in direction. The second method depends upon the change of resistance of a bismuth spiral when placed in a magnetic field (see page 48, " Principles of Electrical Engineering ").

The ballistic method has the advantage that it can be used for the measurement of fields of very low intensity (e.g., the Earth's field) as well as of fields of the greatest intensity which occur in practice. The disadvantages of this method is that the setting up of the apparatus and the taking of the measurements involves a good deal of time, and the evaluation of the results necessitates a considerable amount of calculation.

The second method, based on the change of the electrical resistance of a bismuth spiral when placed in a magnetic field, has the advantages of the rapidity and ease with which the tests can be made. The greatest disadvantage, however, is the much smaller degree of accuracy and reliability which can be obtained by this method and for this reason the practical applications are at present very limited.

The improved method of procedure as described by G. S. Smith (*Electrical Engineering* (New York), 1937, p. 441) has greatly extended the usefulness of the test and has extended the range of its application, particularly as regards the measurement of fields of low intensity. Fig. 21 shows the general arrangement of circuit connections. Two bismuth spirals 1 and 3 are used and these are placed respectively in two opposite arms of the Wheatstone Bridge. In order to obtain the full benefit of the increased sensitivity of the method, it is necessary so to construct the Bridge that sources of disturbance shall be eliminated, the chief cause of disturbance being of thermal origin. That is to say, high values of the thermo-e.m.f. will be produced at the junction of bismuth and any other metal (see

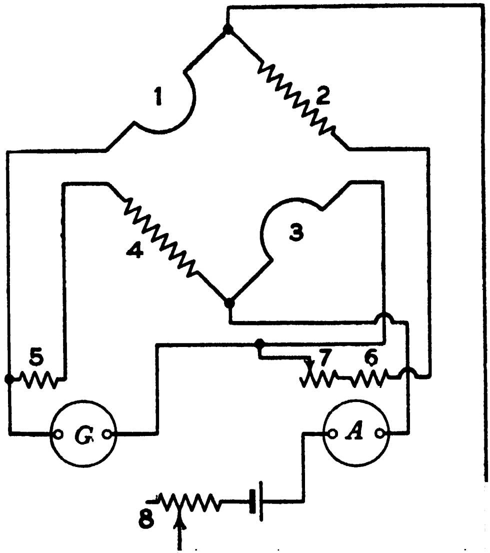

Fig. 21

"Principles", page 159) and special means must be provided to ensure that all such junctions are maintained at the same temperature.

Changes of resistance of the bismuth spiral due to the heating effect of the Bridge current or due to changes of atmospheric temperature have to be compensated and for this purpose the two other arms (2 and 4) are of nickel spirals of which the temperature coefficient is greater than that of bismuth (see page 45, "Principles") and the absolute value of the resistance changes of these arms of the Bridge are always equal to those of the corresponding bismuth spirals. For the purpose of ensuring an exact elimination of temperature change of resistances, manganin resistances (5, 6 and 7) are connected to both pairs of Bridge arms, one of these resistances being adjustable as shown in Fig. 21. In order to ensure an equalization of heating over all the arms of the Bridge the bismuth and nickel spirals are inter-wound.

15.—What is the "Hall Effect" and how may it be applied to the measurement of the strength of a magnetic field.

ANSWER.—In the year 1880, E. H. Hall published an account of his discovery "On a new action of magnetism on a permanent electric current," and this may be briefly described as follows. Suppose a rectangular metal plate is carrying a current in the direction of its length, then if a magnetic field is impressed on the plate in a direction at right-angles to its surface, an e.m.f. will be developed in the plate in a direction perpendicular to the current flow.

In Fig. 22 *AaBb* is the rectangular metal plate and a current of *I* amperes is flowing in the direction from *A* to *B*. If a stationary magnetic field of intensity, *H* oersted is directed at right-angles to the surface of the plate, then an e.m.f. of *E* volts will appear between the points *a* and *b* of the plate. The mutual relationship between the directions of the current flow, *I*, the magnetic field *H* and the developed e.m.f. *E* are shown by the respective arrows in Fig. 22. It will be seen, therefore, that these three arrows are mutually at right-angles and can be represented by the left-hand if the middle-finger, the forefinger, and the thumb are placed mutually at right-angles and are taken to represent the respective arrows in Fig. 22. viz.,

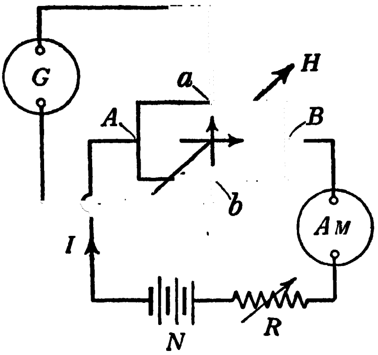

Fig. 22

The forefinger points in the direction of *H*
,, middle-finger ,, ,, ,, ,, ,, *I*
,, thumb ,, ,, ,, ,, ,, *E*

and this rule should be compared with the left-hand rule given in Fig. 37, page 234 of "Principles of Electrical Engineering."

If either the direction of the current flow or the direction of the magnetic field is reversed in sign, the direction of the e.m.f. also becomes reversed.

Hall found from his first experiments that the numerical relationship between the individual quantities is given by the following expression :

$$E = K\frac{H \times I}{d} \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (1)$$

where K is the " Hall Effect Constant," and d is the breadth of the metal plate, that is the width ab in Fig. 22. Further investigation showed that the full effect is obtained with the plate ratio :

$$\frac{\text{breadth}}{\text{length}} = \frac{2}{3}$$

The constant K in the expression (1) has different values and different signs for different metals. The most pronounced Hall Effect is obtained with a plate of *bismuth.*

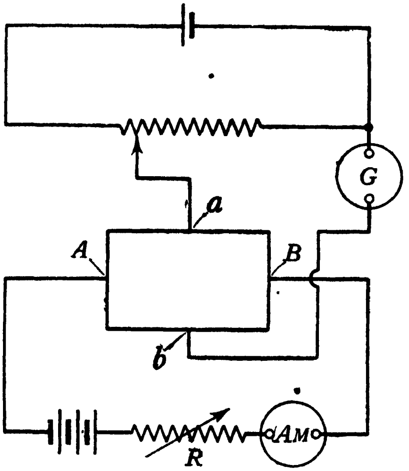

Fig. 23

For the application of the phenomenon to the measurement of the strength of a magnetic field, the circuit system of Fig. 23 may be used, where $AaBb$ represents the bismuth metal plate. The measurement of the e.m.f. E, which is developed between the points a and b, is made by balancing this e.m.f. against the p.d. of the slide-wire so that the galvanometer G reads zero.

Chapter VIII

ELECTRO-MAGNETISM

1.—The pole shoe of a direct current dynamo is shaped as shown in Fig. 1. Explain how the total flux per pole crossing the air-gap per pole may be determined.

Answer.—In Example 5, Chapter VII, the change in direction of the lines of magnetic induction as they pass from one medium to another has been considered. In Fig. 1 are now shown the lines of magnetic induction between the pole face and the armature surface of a dynamo machine, the pole faces being "shaped" in the neighbourhood of the two sides as seen in Fig. 1. The induction B_0 gauss will, therefore, be constant over the width of pole face b_0 cm., for which the air-gap length δ cm. is constant, whilst in the neighbourhood of each of the sides of the pole-shoe the tubes of induction will spread, and the flux density in each of these regions will become correspondingly reduced. The distribution of induction density under the pole face will be such that the total energy of the magnetic field in this space will be a minimum.

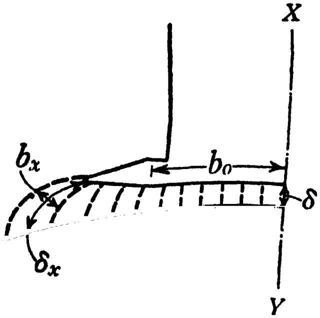

Fig. 1

The pole-shoe face and the armature surface are each equi-potential magnetic surfaces, so that the magneto-motive force across all the tubes will have the same value. For the pole breadth b_0 the relationship between flux density and m.m.f. is—

$$\text{m.m.f.} = B_0\,\delta$$

and the flux which crosses the gap under the central part of the pole face, that is, over the breadth b_0 cm., will be—

$$\Phi_0 = B_0\,b_0\,l$$

where l cm. is the axial length of the pole and armature. That is,

$$\Phi_0 \quad \text{m.m.f.}\left(\frac{b_0 l}{\delta}\right)$$

For a tube of flux in the neighbourhood of the edge of the pole shoe, the magnetic conductivity is—

$$\lambda_x = \frac{b_x}{\delta_x}\,l$$

so that the magnetic flux comprised by this tube will be—

$$\triangle\,\Phi_x = \text{m.m.f.} \times \lambda_x = B_0\,\delta\,\frac{b_x}{\delta_x}\,l$$

The total flux for both the shaped sides of the pole shoe will then be—

$$2\Phi_x = 2\Sigma\left(B_0\,\delta l\frac{b_x}{\delta_x}\right) = 2B_0\,\delta l\,\Sigma\frac{b_x}{\delta_x}$$

and consequently, the total flux per pole which crosses the gap will be—

$$\Phi = \Phi_0 + 2\Phi_x = B_0\,2\,b_0\,l + 2B_0\,l\,\delta\,\Sigma\frac{b_x}{\delta_x} = B_0\,l\,b_i$$

where $$b_i = 2b_0 + 2\delta\,\Sigma\frac{b_x}{\delta_x}$$

and is termed the " ideal pole breadth." Hence, the total flux Φ per pole is the same as that which would be the uniform gap of δcm. if the pole breadth were b_i cm., the flux density everywhere in the gap having the constant value of B_0 gauss.

2.—*A magnetic separator, capable of dealing with large blocks of iron, comprises a " magnetic pulley " driven by a belt on which the material to be separated travels. The pulley is magnetised by means of an exciting coil as shown diagrammatically in Fig. 2. Find what magnetic pull is exerted on the block when it is symmetrically placed over the axis of the pulley as shown in Fig. 3. The diameter of the pulley is 36 inches and the belt is $\frac{5}{8}$-inch thick. The surface of the block is $6\frac{1}{4} \times 24$ inches, the effective length in the direction of the pulley axis being 12 inches. The exciting ampere-turns of the magnetising coil is 12,000.*

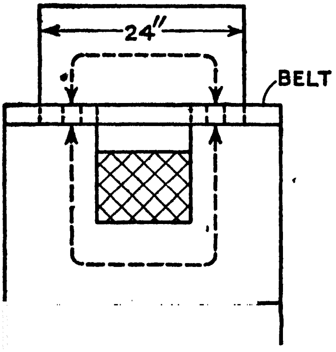

Fig. 2

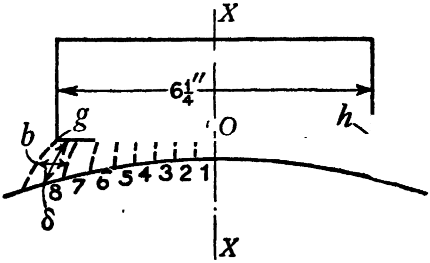

Fig. 3

Answer.—In Fig. 2 the tubes of force between the block and the magnetic pulley are shown, as before, the guiding principle in drawing these lines of force being that they enter and leave the magnetised surfaces practically at right angles to the surface. Since the system is symmetrical about the median line XX it is only necessary to calculate the pull due to one-half the total number of tubes and then multiplying the result by 2 to obtain the total pull. Each tube of force will cross the gap twice (see Fig. 1), so that the mean value of the flux density in any one tube will be given by the equation—

$$\frac{4\pi}{10}\text{ ampere-turns} = H2\,\delta_x \qquad 2B_x\,\delta_x$$

since $\mu=1$ and $B=H$ for the air-gap. In this equation, δ_x cm. is the mean length of gap in any one tube. Then—

$$B_x = \frac{0{\cdot}63}{\delta_x}\,12{,}000 = \frac{7{,}560}{\delta_x}\text{ gauss.}$$

The mean breadth of each tube is 1 cm. and the effective length in a direction perpendicular to the plane of the paper (see Fig. 2) is 12 inches, that is, approximately 30 cm., so that the total effective surface of the block for each tube is about 30 × 1 = 30 sq. cm. The pull, therefore, which is due to each tube of the half-surface Og in Fig. 3 will be given by the expression—

$$P = 0{\cdot}042\left(\frac{B_x}{1{,}000}\right)^2 S = 1{\cdot}26\left(\frac{B_x}{1{,}000}\right)^2\text{ kg. weight}$$

where B_x gauss is the mean value of the flux density in the tube considered. The calculation can then be arranged in tabular form as follows—

Tube Number	Mean Length of Gap : δ cm.	Mean Flux Density : $B = \dfrac{7{,}560}{\delta}$ gauss	Magnetic Pull $1{\cdot}26\left(\dfrac{B}{1{,}000}\right)^2$ kg. weight
1	1·6	4,750	28·5
2	1·6	4,750	28·5
3	1·6	4,750	28·5
4	1·7	4,470	25·5
5	1·8	4,220	22·5
6	1·9	4,000	20·2
7	2·1	3,630	16·5
8	2·4	3,160	12·7

Total : 183 kg. weight.

The same pull will be produced on the other half of the block surface Og, so that the total pull on the block will be 366 kg. weight.

3.—A current of 8 amperes is flowing in a long straight wire. What is the force on a unit magnetic pole placed at a distance of 7 cm. from the axis of the wire? What will be the radius of a circular coil of one turn which will produce the same force at the centre of the coil for the same current of 8 amperes in the coil?

ANSWER.—The force at a point distant 7 cm. from a long straight wire which is carrying a current of 8 amperes is—

$$F = \frac{2i}{10d} = \frac{2 \times 8}{10 \times 7} = 0{\cdot}23 \text{ dynes.}$$

The radius r cm. of a circular coil of 1 turn which will produce the same force at the centre of the coil for the same current of 8 amperes will be given by the expression—

$$F = 0{\cdot}23 = \frac{2\pi i}{10r} = \frac{2\pi 8}{10 \times r}$$

so that

$$r = \frac{16\pi}{2{\cdot}3} = 22 \text{ cm.}$$

4.—A lifting electromagnet and armature of soft iron are shown in Fig. 4, the air-gap δ being 2 mm. and the total length of the magnetic circuit in the iron being 130 cm. The section of the iron is 9 × 9 cm. and the permeability is 2,500. What flux density in the air-gap will be required to hold a load of 1 metric ton, and what number of ampere-turns must be provided to produce this flux density?

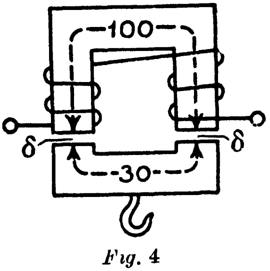

Fig. 4

ANSWER.—The pull exerted on the armature by the electromagnet is

$$P = 0{\cdot}042 \left(\frac{B}{1{,}000}\right)^2 S \text{ kg. weight}$$

where $S = 2 \times 9 \times 9 = 162$ sq. cms. $P = 1{,}000$ kg.

so that

$$\left(\frac{B}{1{,}000}\right)^2 = \frac{1{,}000}{0{\cdot}042 \times 162} = \frac{1{,}000}{6{\cdot}8} = 147$$

$$\frac{B}{1{,}000} = 12{\cdot}1 : \underline{B = 12{,}100 \text{ gauss.}}$$

Then, $$\frac{4\pi}{10} \text{ ampere-turns} = H_1 l_1 + H_2 l_2$$

where H_1 oersted and $l_1 = 2\delta$ cm. refer to the air-gap
where H_2 oersted and l_2 cm. refer to the magnetic path in the iron.

Hence $$\frac{4\pi}{10} \text{ ampere-turns} = 12{,}100 \times 0{\cdot}4 + \frac{12{,}100}{2{,}500} \times 130$$
$$= 4{,}840 + 630 = 5{,}470$$

that is $$\text{ampere turns} = 0{\cdot}8 \times 5{,}470 = \underline{4{,}376}$$

5.—In order to generate the required direct current e.m.f., the flux in the armature of the two-pole dynamo of Fig. 5 must be 5·75 × 10⁶ c.g.s. lines and the magnetic permeability is $\mu = 3{,}000$. The magnetic

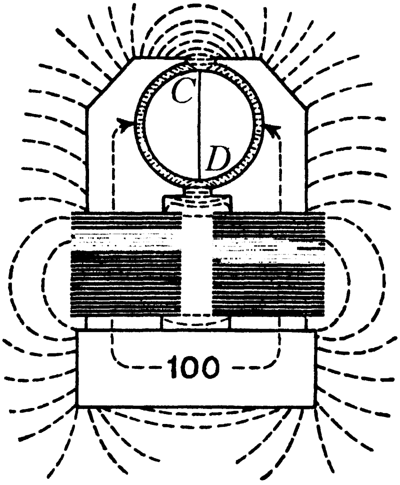

Fig 5

leakage factor is 1.2: the mean length of path in the poles and yoke is 100 cm. and the cross-section is 650 sq. cm.: the length of the single air-gap is 1 cm. and the cross-section is 1,300 sq. cm.: the length of path in the armature is 15 cm. and the cross-section is 500 sq. cm. Find what exciting ampere-turns will be required.

ANSWER.—Since the armature flux is $5{\cdot}75 \times 10^6$ and the leakage factor is 1·2 the flux in the poles and yoke will be $1{\cdot}2 \times 5{\cdot}75 \times 10^6 = 6{\cdot}9 \times 10^6$ c.g.s. lines. The following Table then gives the necessary data for the component parts of the magnetic circuit from which the required total number of exciting ampere-turns is found.

In this Table the symbol *aw* is used to denote ampere-turns per cm. length of path and *AW* to denote the total number of ampere-turns.

Component	Φ	S sq. cm.	B gauss	$H = \frac{B}{\mu}$ oersted	aw $= 0{\cdot}8H$	l cm.	AW
Poles and Yoke ..	$6{\cdot}9 \times 10^6$	650	10,600	3·53	2·82	100	282
Air-gap ..	$5{\cdot}75 \times 10^6$	1,300	4,400	4,400	3,520	$2\delta = 2$cm.	7,140
Armature	$5{\cdot}75 \times 10^6$	550	10,400	3·47	2·78	15	42

Total : 7,464

6.—A short-circuited circular coil of 150 turns has a mean diameter of 25·5 cm. and is arranged so that it can be rotated about a vertical axis. The coil is set with its plane perpendicular to the horizontal component of the earth's magnetic field, the intensity of which is 0·2 oersted. If the resistance of the coil is 20 ohms, find what quantity of electricity will be set in motion when the coil is turned through 180°.

ANSWER.—The mean area of the coil is $\frac{\pi}{4} \times 25{\cdot}5^2 = 510$ sq. cm.

The magnetic flux embraced by the coil in its initial position is $\Phi = 0{\cdot}2 \times 510 = 102$ c.g.s. lines (maxwells), so that the flux-linkages will then be $\Phi w = 102 \times 150 = 15{,}300$.

When the coil has been turned through 180° the flux-linkages of the coil will have become reversed so that the change of flux-linkages due to this half-rotation will be $2\Phi w$ 30,600. The quantity of electricity set in motion by turning the coil through 180° will therefore be—

$$Q = \frac{2\,\Phi w}{r\,10^8} = \frac{30{,}600}{20 \times 10^8} = 15{\cdot}3 \times 10^{-6} \text{ coulomb.}$$

7.—Show how the mutual inductance of two coils can be measured by means of a ballistic galvanometer test.

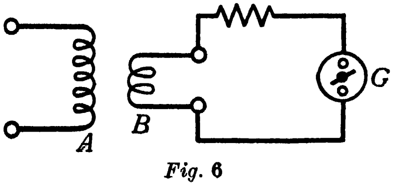

Fig. 6

ANSWER.—In Fig. 6 *A* is the primary and *B* the secondary of two mutually inductive coils and it is required to measure the coefficient of mutual induction *M* henry. Let a current of *i* amperes flow in the coil *A* and let *B* be connected to a ballistic galvanometer through a suitable resistance. The flux-linkages of coil B due to a current of i amperes in coil A is then,

$$10^8\, Mi \text{ c.g.s. lines.}$$

If Q coulombs is the quantity of electricity which is discharged through the ballistic galvanometer when the current circuit of coil A is broken then,

$$Q = \frac{10^8\, Mi}{r \times 10^8} = \frac{Mi}{r} \text{ coulombs}$$

where r ohms is the total resistance in the circuit of the coil B and the galvanometer. Hence,

$$M = \frac{Qr}{i} \text{ henry.}$$

8.—Two coils, 1 and 2, are in inductive relationship to one another, the resistance of each coil being negligible. If a d.c. pressure of 10 volts is applied to coil 1, obtain a graph showing the current in each coil and the stored electromagnetic energy of the system as a function of the time, when $L_1 = 1$ henry: $L_2 = 2$ henry: $M = 1.2$ henry.

ANSWER.—The two coils are shown in mutually inductive relationship in Fig. 7, and the following two simultaneous equations define the current in the respective circuit as functions of the time, viz.:

$$\left.\begin{aligned} V &= L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \\ O &= L_2\frac{di_2}{dt} + M\frac{di_1}{dt} \end{aligned}\right\}$$

eliminating i_2 between these equations gives

$$V = L_1\frac{di_1}{dt} - \frac{M^2}{L_2}\frac{di_1}{dt}$$

or

$$\frac{di_1}{dt} = \frac{V}{L_1 - \dfrac{M^2}{L_2}}$$

from which it follows that

$$i_1 = \left(\frac{V}{L_1 - \dfrac{M^2}{L_2}}\right) t$$

Similarly, since

$$\frac{di_2}{dt} = -\frac{M}{L_2}\frac{di_1}{dt}$$

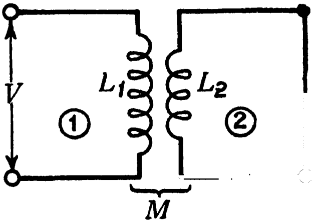

Fig. 7

then

$$i_2 = -\frac{M}{L_2}\left\{\frac{V}{L_1 - \frac{M^2}{L_2}}\right\} t = -\frac{M}{L_2} i_1$$

The total stored electromagnetic energy in the system at any time t is then

$$U = \int_0^t V i_1 dt = \int_0^t \frac{V^2}{L_1 - \frac{M^2}{L_2}} t dt = \frac{1}{2} \frac{V^2}{L_1 - \frac{M^2}{L_2}} t^2 \text{ joules}$$

that is,

$$U = \frac{1}{2} i_1^2 \left(L_1 - \frac{M^2}{L_2}\right) = \frac{1}{2} i_1^2 L_1 - \frac{1}{2} i_2^2 L_2$$

so that,

$$U = \frac{1}{2} i_1^2 L_1 + \frac{1}{2} i_2^2 L_2 - i_2^2 L_2$$

and hence,

$$U = \frac{1}{2} i_1^2 L_1 + \frac{1}{2} i_2^2 L_2 - i_1 i_2 M \text{ joules}$$

Inserting the numerical values

$$L_1 = 1 \text{ henry} : L_2 = 2 \text{ henry} : M = 1{\cdot}2 \text{ henry}$$

$$V = 10 \text{ volts. Then } L_1 - \frac{M^2}{L_2} = 0{\cdot}28 \text{ henry}$$

$$i_1 = 35{\cdot}7t \text{ amperes} : i_2 = -21{\cdot}4t \text{ amperes}$$

$$U = 179t^2 \text{ joules}$$

These results are shown by the graphs of Fig. 8.

An interesting demonstration of the growth of current in circuits which are electromagnetically coupled, and of which one is connected to a source of d.c. pressure, can be obtained with the arrangement shown in Fig. 9.

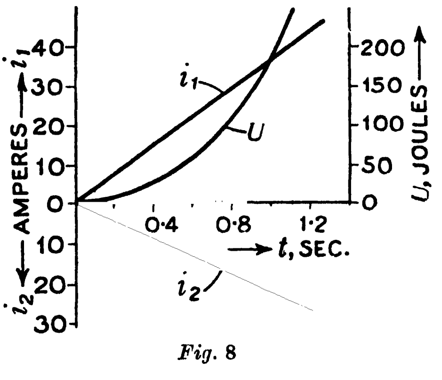

Fig. 8

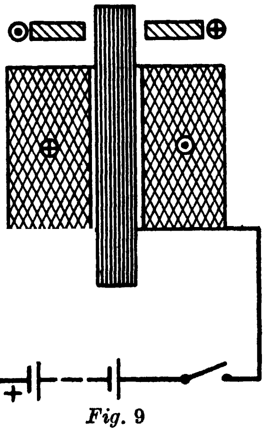

Fig. 9

A solenoid is provided with a core consisting of a bundle of iron wires and an aluminium ring is placed over the projecting end of the iron core. The solenoid is connected through a switch to a battery, and it will be found that at the moment of closing the switch the ring will be shot up into the air. This is an example of electromagnetic repulsion, and can be accounted for as follows:

When the switch is closed the current in the solenoid winding and, consequently, the flux in the iron core, grows in accordance with a logarithmic curve. This increasing flux induces a current in the aluminium ring, the direction of the current being such that it opposes the rising flux in the solenoid core (Lenz's Law). A force therefore acts on the ring tending to drive off the core of the solenoid, and thus prevent it from becoming linked by the rising flux in the core.

These conditions are fundamentally the same as those which are defined by the graphs of Fig. 8.

9.—A two-pole short-circuit occurs at a power station which is operating in parallel with a number of other stations on 10 kV. bus bars. The short-circuit current reaches a peak value of 93,500 amperes. If the length of conductor between two supports is 160 cm. and the mean distance between two neighbouring conductors is 40 cm., find what mechanical force will be developed between two conductors.

ANSWER.—The mechanical force between the conductors is given by the expression

$$P = 2{\cdot}04\, I^2 \frac{l}{D} 10^{-8} \text{ kg.}$$

Inserting the numerical values given in the question, viz., $l = 160$ cm., $D = 40$ cms., gives

$$2{\cdot}04(93{,}500)^2 \times \frac{160}{40} \times 10^{-8} = 720 \text{ kg. weight}$$

and the supporting structure for the conductors must be able to withstand this force.

10.—Obtain the relationship between torque and speed for a direct current series wound traction motor.

ANSWER.—The connections for the series type of traction motor are shown diagrammatically in Fig. 10, in which A is the armature and M the field winding. The direction of rotation of the armature will remain the same whether polarity of the supply terminals is as shown, or whether it is reversed, since reversing the polarity will reverse the current in both field and armature winding, this leaving the direction of the torque unaltered. It follows from this that the torque on the armature will be continuously in the same direction whether the supply is single-phase alternating current or direct current. When a.c. is used the whole of the magnetic circuit must be laminated, that is, the field structure as well as the armature core.

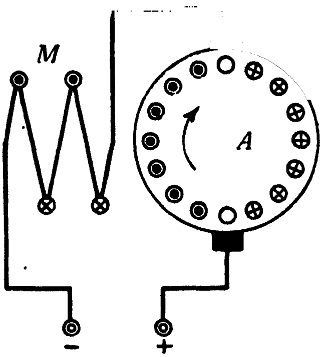

Fig. 10

Suppose the motor is supplied from a d.c. source of constant terminal pressure V volts and let R ohms be the resistance of the field and armature windings, then the induced e.m.f. E_a will be given by the equation

$$E_a = V - I.R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If Φ is the magnetic flux per pole and n r.p.s. is the speed of rotation of the armature, then

$$E_a = k\, \Phi\, n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where k is a constant of the machine.

The mechanical power developed by the motor will then be E_aI watts so that if the torque is τ kg.-m. units, then,

$$E_a\, I_a = 2\pi \times 9{\cdot}81\, \tau.n$$

or, substituting for E_a the expression given in (2), it is seen that

$$k\, \Phi\, nI = 2\pi n\, 9{\cdot}81 \tau.n$$

that is

$$\tau = 0{\cdot}016 k\, \Phi\, I \text{ kg.m.}$$

For railway work the " tractive effort " F is usually specified instead of the torque, viz., the tractive effort is the force in kg. at the rim of the driving wheels, so that (see Fig. 11),

$$\text{Torque in kg.-m.} = (\text{tractive effort}) \times \frac{D}{2}$$

where D metres is the diameter of the driving wheels.

The speed of the train is given by

$$v = \pi\, D\, N/60 \text{ metres per sec.}$$

where N revs. per min. is the speed of the driving wheels; the power developed by the motor is then,

$$W = v\, F\, 9{\cdot}81 \times 10^{-3} \text{ kw.}$$

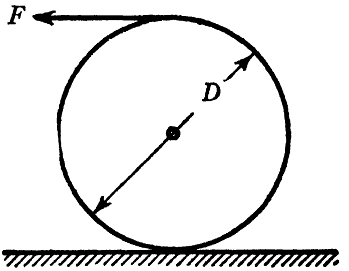

Fig. 11

11.—The switch shown in Fig. 12 comprises two surfaces in contact, the process of opening the switch consisting in sliding these two surfaces so that the area of contact gradually becomes zero. Obtain the relationships between contact current and time and contact p.d. and time.

ANSWER.—If S is the full contact area and if y second is the time taken for operating the switch, then the surface contact at any moment t will be

$$S\left(1-\frac{t}{y}\right)$$

If r ohms is the contact resistance for the full surface S then at the moment t the resistance will be,

$$r\frac{S}{S\left(1-\frac{t}{y}\right)}=r\frac{y}{y-t}\text{ ohms.}$$

The equation defining the current at any moment t is then

$$L\frac{di}{dt}+Ri+r\frac{y}{y-t}i=V\ldots\ldots\ldots\ldots(1)$$

The complete solution of this equation involves complicated processes and for practical purposes (e.g., a study of resistance commutation) it is sufficient to examine the conditions which obtain when the switch is about to open. At the commencement of the switching operation the contact resistance r will be small and the p.d. v across the contact will also be small. When approaching the moment at which the contacts will separate the resistance increases rapidly and the current falls to zero as shown in the diagram of Fig. 13, the pressure v being given by the equation

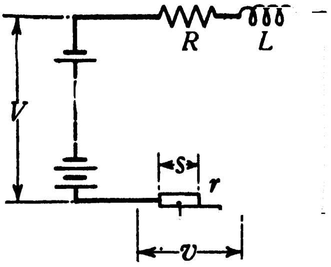

Fig. 12

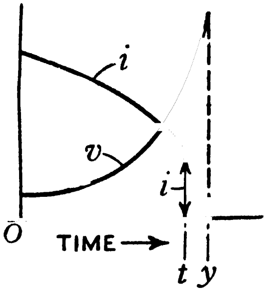

Fig. 13

$$v=\frac{ry}{y-t}i\ldots\ldots\ldots\ldots(2)$$

Near the moment of opening, the rate of fall of the current can be expressed by the relationship

$$-\frac{di}{dt}=\frac{i}{y-t}$$

and inserting this value in equation (1) gives

$$-L\left(\frac{i}{y-t}\right)+Ri+ry\left(\frac{i}{y-t}\right)=V\ldots\ldots\ldots\ldots(3)$$

As t approaches y the current becomes small and the ratio $\frac{y}{y-t}$ becomes large so that the term Ri becomes negligible and hence, from equations (1) and (3)

$$-L\frac{v}{ry} + v = V$$

or

$$v\Big]_{t=y} = \frac{VL}{1 - \frac{L}{ry}}$$

It is seen, therefore, that the conditions which must hold if the p.d. across the switch at the moment of opening is not to become indefinitely large is that

$$\frac{L}{ry} > 1 \quad \text{that is} \quad \boxed{y > \frac{L}{r}}$$

12.—*An electron is moving in a steady homogeneous electric field and a steady homogeneous magnetic field, these two fields acting in directions which are mutually at right-angles. Obtain the equations which define the motion of the electron.*

ANSWER.—Suppose, for example, that the electron is emitted from a hot filament and that it is moving in the plane of the x, y co-ordinate axes as shown in Fig. 14, where P shows the position of the electron at any moment t. The steady homogeneous electric field of strength E electromagnetic units is acting in the plane of the x, y co-ordinate axes, and the steady homogeneous magnetic field of strength H oersted is acting in the direction of the z co-ordinate axis, that is, at right-angles to the direction of E as is shown in Fig. 14.

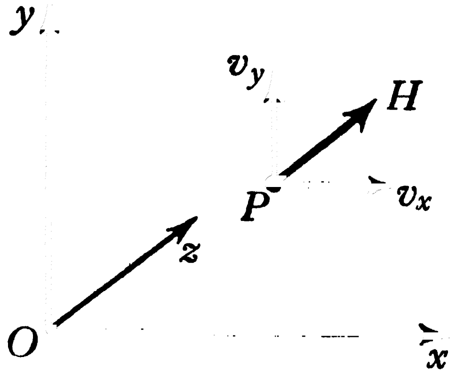

Fig. 14

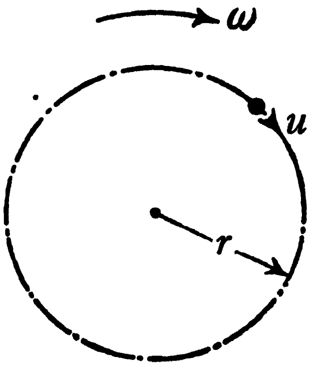

Fig. 15

Now an electron of charge e electromagnetic units, moving with a velocity of v cm. per sec. forms an electric current of strength.*

$$C_x = -\,e.\,v_x \text{ electromagnetic units} \;\ldots\ldots\ldots\ldots (1)$$

If v_x cm. per sec. is the velocity component in the x direction, the electron will experience forces as follows :

(i) A force in the direction Oy due to the magnetic field H, viz.,

$$F_y]_H = -H{\cdot}C_x = -H.e.v_y \text{ dynes} \;\ldots\ldots\ldots\ldots (2)$$

(ii) A force in the direction Oy due to the electric field E, viz.,

$$F_y]_E = E.e \text{ dynes} \;\ldots\ldots\ldots\ldots (3)$$

Hence the mass-acceleration of the electron in the direction Oy will be

$$m.\frac{dv_y}{dt} = E.e - H.e.v_x \;\ldots\ldots\ldots\ldots (4)$$

Similarly, the electron will acquire a mass-acceleration in the direction Ox, viz.,

$$m\,\frac{dv_x}{dt} = H.e.v_y \;\ldots\ldots\ldots\ldots (5)$$

Solving equations (4) and (5) for v_x and v_y respectively, gives

$$\left.\begin{array}{l} v_x = u \sin\,[\omega\,(t-t_o)] + \dfrac{E}{H} \ldots\ldots (a) \\ v_y = u \cos\,[\omega\,(t-t_o)] \ldots\ldots\ldots\ldots (b) \end{array}\right\} \;\ldots\ldots\ldots\ldots (6)$$

where $\frac{E}{H}$ is a velocity of translation of constant magnitude in the direction Ox. Apart from this uniform velocity of translation, the equations (6) define a circular motion of linear velocity u cm. per sec. and angular velocity

$$\omega = \frac{e.\,H}{m} \text{ radians per sec.} \;\ldots\ldots\ldots\ldots (7)$$

as shown in Fig. 15, the initial conditions being defined by t_o.

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* It is to be observed that the standard definition of the direction of a current is the direction in which positive electric charges are moving, so that an electron moving in the direction v_x forms an electric current of which the direction is given by $-\,v_x$.

13.—Derive the equations which define the path of the electron motion of Question 12 for the following initial conditions, viz.,

$$t = o : v_x = o : v_y = o$$

ANSWER.—Choosing the origin of co-ordinates such that x=o : y=o : when t=o, the general equations for the motion of the electron for the prescribed initial conditions may be obtained.

The general equation for the motion of the electron as already derived in Example 12 are,

$$\left.\begin{aligned} v_x &= u \sin(\omega t - \omega t_o) + \frac{E}{H} \\ v_y &= u \cos(\omega t - \omega t_o) \end{aligned}\right\} \dots\dots\dots\dots (1)$$

When the initial conditions are inserted in these equations, it is found that the motion of the electron is defined by the equations,

$$\left.\begin{aligned} v_x &= \frac{E}{H}(1 - \cos \omega t) \\ v_y &= \frac{E}{H} \sin \omega t \end{aligned}\right\} \dots\dots\dots\dots (2)$$

The path of the electron is then defined by the relationships,

$$\left.\begin{aligned} x &= \int v_x \, dt = \frac{E}{H} t - \frac{E}{H\omega} \sin \omega t + C \\ y &= \int v_y \, dt = -\frac{E}{H\omega} \cos \omega t + K \end{aligned}\right\} \dots\dots\dots\dots (3)$$

where K and C are arbitrary constants which again are defined by the initial conditions. That is to say, inserting in equations (3) the conditions,

$$t = o : x = o : y = o,$$

it is found that $C = o : K = \dfrac{E}{H\omega}$

and consequently, the path of the particle is found to be defined by the equations,

$$\left.\begin{aligned} x &= \frac{E}{H\omega}(\omega t - \sin \omega t) \\ y &= \frac{E}{H\omega}(1 - \cos \omega t) \end{aligned}\right\} \dots\dots\dots\dots (4)$$

These are the standard equations for a cycloid*, for which the origin of co-ordinates lies on the base on which the generating circle rolls as shown in Fig. 16. The radius of the generating circle is at once obtained from the equations (4), viz.,

$$r = \frac{E}{H.\omega} \quad \text{..........} \quad (5)$$

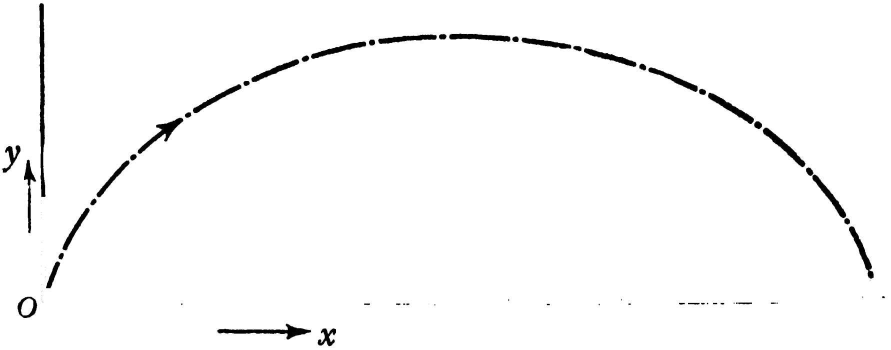

Fig. 16

The principle which is illustrated by this problem is applied in the operation of the " Magnetron " valve for generating ultra-high-frequency oscillations (see also : T.F.W. : " Engineering," Feb. 8, 1946, p. 126).

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* See, for example, Lamb's " Calculus."

Chapter IX

ALTERNATING CURRENTS : THE USE OF COMPLEX QUANTITIES

1.—A condenser of 3μF capacitance is placed in parallel with a resistance of 150 ohms and the parallel system is connected to an a.c. supply of 600 r.m.s. volts at 500 frequency. What is the power factor and the magnitude of the current which is supplied by the mains ? If the resistance and capacitance are now placed in series across the same supply terminals, what will be the current and power factor ?

ANSWER.—(i) *Current in the Parallel System of Fig.* 1.

$$I_c = j\omega CV = j3{,}140 \times 3 \times 10^{-6} \times 600 = j\ 5{\cdot}68 \text{ amperes.}$$

$$I_r = \frac{V}{R} = \frac{600}{150} = 4 \text{ amperes.}$$

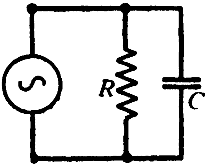


Fig. 1

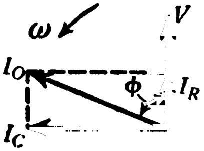


Fig. 1*a*

The vector sum of these two currents is (Fig. 1*a*), $I_o = \sqrt{5{\cdot}68^2 + 4^2}$ $= 6{\cdot}95$ amperes : $\tan\phi = \dfrac{I_c}{I_r}$

that is

$$\tan\phi = \frac{\omega CV}{V/R} = \omega CR = \frac{3{,}140 \times 3 \times 150}{10^6} = 1{\cdot}4$$

$$\phi = 55^c : \cos\phi = 0{\cdot}573.$$

(ii) *Current in the Series System of Fig.* 2.

$$I_o = \frac{V}{R - j\dfrac{1}{\omega C}} = V\frac{R + j\dfrac{1}{\omega C}}{R^2 + \left(\dfrac{1}{\omega C}\right)^2}$$

$$\frac{1}{\omega C} = \frac{10^6}{3{,}140 \times 3} = 106\ \Omega : I_o = V\frac{105 + j106}{150^2 + 106^2}$$

so that

$$I_o = 2{\cdot}67 + j\,1{\cdot}89 \text{ amperes.}$$

Magnitude of current is $I_o = \sqrt{2{\cdot}67^2 + 1{\cdot}89^2} = 3{\cdot}27$ amperes and

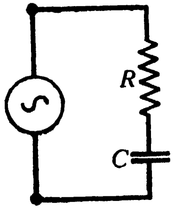


Fig. 2

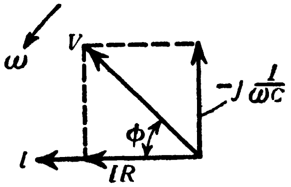


Fig. 2a

(see Fig. 2*a*) $\tan\phi = \dfrac{\dfrac{I}{\omega C}}{IR} = \dfrac{1}{\omega CR} = 0{\cdot}7$

so that

$$\phi = 35 \;:\; \cos\phi = 0{\cdot}819.$$

2.—In order to protect a group of relatively small power cables from a short-circuit which might produce a destructive effect, a choking coil can be arranged between the generator and the cables as shown in Fig. 3.

A water-driven three-phase power station of 150 MVA. capacity at 6.6/$\sqrt{3}$ kV. per phase supplies a nitrogen plant by means of cables. The reactance of the station is 15 per cent. and the reactance of the protective choke coil is 8 per cent. Find the value of the short-circuit current.

What will be the short-circuit current if there is no protective choking coil? What is the ratio of the consequent mechanical stress in the two cases?

ANSWER.—The circuit system is shown in Fig. 3. Pressure per phase

$$\frac{6600}{\sqrt{3}} = 3{,}820 \text{ volts.}$$

150 MVA. = 150×10^6 VA.

Normal current per phase $= \dfrac{150}{3} \times \dfrac{10^6}{3{,}820} = 13{,}100$ amperes.

Station reactance per phase X_s is given by $X_s \times 13{,}100 = 0{\cdot}15 \times 3{,}820$: so that $X_s = 0.044\,\Omega$.

Choke coil reactance per phase X_c is given by $X_c \times 13{,}100 = 0{\cdot}08 \times 3{,}820$: so that $X_c = 0{\cdot}0234\,\Omega$.

Total reactance per phase $= 0{\cdot}0674\,\Omega$.

Short-circuit current per phase is $I_{s.c} = \frac{3,820}{0{\cdot}0674} = 57,000$ amperes.

When no choke coil is used, the total reactance per phase is $0.044\ \Omega$, so that the short-circuit current per phase will then be

$$I_{s.c} = \frac{3,820}{0{\cdot}044} = 87,000 \text{ amperes.}$$

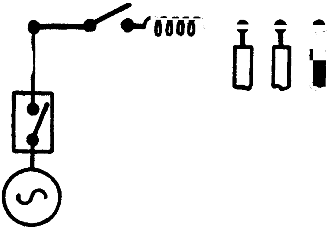


Fig. 3

Since the mechanical force developed by a current-carrying conductor is proportional to the square of the current, the relationship of the mechanical forces developed in the foregoing cases is $\left(\frac{87,000}{57,000}\right)^2 = 2{\cdot}3$.

3.—A laminated cylindrical iron ring of 45 sq. cm. cross-section is uniformly wound with a coil of 100 turns, and an alternating p.d. $v = V_m \cos \omega t$ is applied to the terminals of the coil. If the resistance of the winding is negligibly small, obtain an expression for the magnetic induction B as a function of the time and discuss the implications of this expression.

ANSWER.—The induced back e.m.f. of the coil is $e = -\frac{w}{10^8}\frac{d\Phi}{dt}$ volts where Φ is the magnetic flux in c.g.s. lines linked with the coil at any moment t and since $v + e = o : v = V_m \sin \omega t = \frac{w}{10^8}\frac{d\Phi}{dt}$ volts so that,

$$d\ \Phi = \frac{10^8}{w}\ V_m\ (\sin \omega t)\ dt$$

and hence,

$$\Phi = -\frac{V_m\, 10^8}{w\omega} \cos \omega t + K \quad \ldots\ldots\ldots\ldots\ldots\ (1)$$

where K is an arbitrary constant depending upon the initial conditions. Two special cases will be examined in what follows.

(*i*) If the switch is closed when $t = o$, that is, when the e.m.f. wave is passing through its zero value, then the initial conditions are, $t = o$: $\Phi = o$: so that $K = \frac{V_m 10^8}{w\omega}$ and consequently,

$$\Phi = \frac{V_m 10^8}{w\omega}(1 - \cos \omega t) = \Phi_m (1 - \cos \omega t)$$

where, $\Phi_m = \frac{V_m 10^8}{w\omega}$ c.g.s. lines (i.e. maxwells).

If $V_m = \sqrt{2} \times 100$ volts : $w = 100$: $\omega = 2\pi f = 2\pi 50 = 314$ then $\Phi_{max} = 0{\cdot}45 \times 10^6$ maxwells $= 0{\cdot}45 \times 10^{-2}$ webers and since the cross-section of the ring is 45 sq. cm. it is seen that

$$B = B_{max}\,(1 - \cos \omega t) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

where $B_{max} = 10{,}000$ gauss.

The values of the induction B gauss as defined by this relationship are shown in Fig. 4, from which it will be seen that the magnetic flux remains wholly *positive*. If the magnetisation curve for the laminated iron ring is as shown in Fig. 5, in which the magnetomotive force is defined by the number of ampere-turns per cm. length of the magnetic circuit, then from the fundamental equation for m.m.f., viz., 1·26 (ampere-turns) $= Hl$ it is seen that 1·26 ampere-turns per cm. $= H$ oersted or : ampere-turns per cm. $= 0{\cdot}8H$. If the length l is 250 cms., then ampere-turns per cm. $= \frac{100i}{250}$, or $i = 2{\cdot}5$ (ampere-turns per cm.). Neglecting hysteresis effects, the current wave which corresponds to the wave of magnetic induction as defined by the equation (2) is shown in Fig. 4, from which it will be seen that the peak value of this current wave is about 900 amperes.

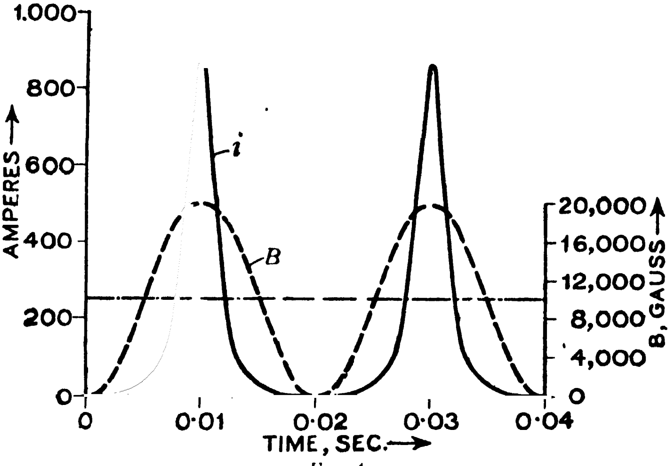


Fig. 4

(*ii*) If the switch is closed at the moment when the applied p.d. is passing through its first positive peak value, then the initial conditions are: $t = \frac{\pi}{2\omega}$: $\Phi = 0$: so that the arbitrary constant in equation (1) is zero and consequently,

$$\Phi = \Phi_m \cos \omega t \text{ and } B = B_{max} \cos \omega t \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

In Fig. 6 this wave of magnetic flux density is shown as well as the corresponding magnetising current wave, from which it will be seen that the peak value of the current wave is now not quite 20 amperes. A comparison of the current waves of Figs. 4 and 6 shows what immensely heavy magnetising currents can be developed if the switch is closed when the applied p.d. is passing through, or near to, its zero value. As already stated, the effect of the resistance of the winding has been neglected in the foregoing treatment, but it is to be observed that, although the resistance may be very small, yet when the current approaches such immensely high peak values as those shown in Fig. 4 the pressure drop due to the resistance will be relatively large and, in this way, the peak current values will be drastically reduced. Even so, the magnetising current may reach values which are dangerously high for the mechanical safety of the coil, it being

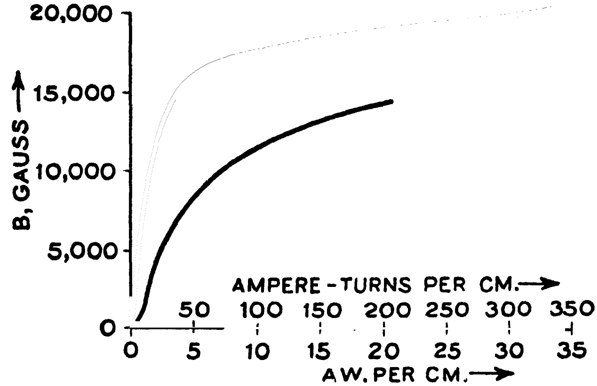


Fig. 5

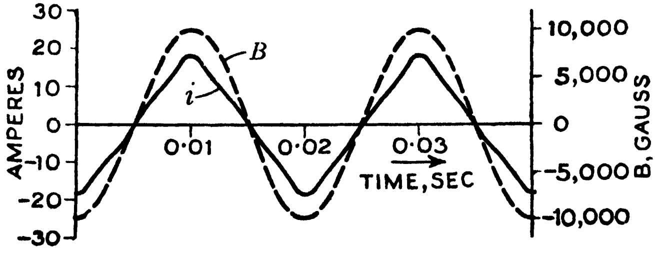


Fig. 6

observed that the mechanical forces due to the current are proportional to the square of the current value. A further effect of the resistance of the winding is that the current wave will not remain wholly positive as in Fig. 4 but will gradually assume the symmetrical current condition shown in Fig. 6, that is, symmetrical about the abscissa axis. The conditions illustrated in Fig. 4 apply to the case of switching in an unloaded transformer and it might quite well happen that the switch closes when the applied p.d. is passing through its zero value, so that if proper precautions are not taken, destructive results may ensue.

One method of eliminating dangerously high current peak values due to this cause is to incorporate in the switch a suitable resistance which is momentarily connected in series with the transformer winding as the switch closes, and when the switch is fully closed the resistance becomes automatically short-circuited.

4.—Show how the magnetising current wave of a transformer may be derived from the hysteresis loop.

Answer.—In Fig. 7 is shown the hysteresis loop for a peak value of the induction density $B = 17{,}000$ gauss, the corresponding value of the magnetising force being $H_m = 45{\cdot}3$ oersted and the corresponding value of the magnetising current being $I_m = 2{\cdot}25$ amperes, thus for $w = 1{,}600$ and for a length of magnetic circuit $l = 100$ cm.

$$1{\cdot}26\, I_m\, 1{,}600 = H_m\, 100$$

$$\text{or } H_m = 20{\cdot}2\, I_m$$

The method by which the magnetising current wave may be derived

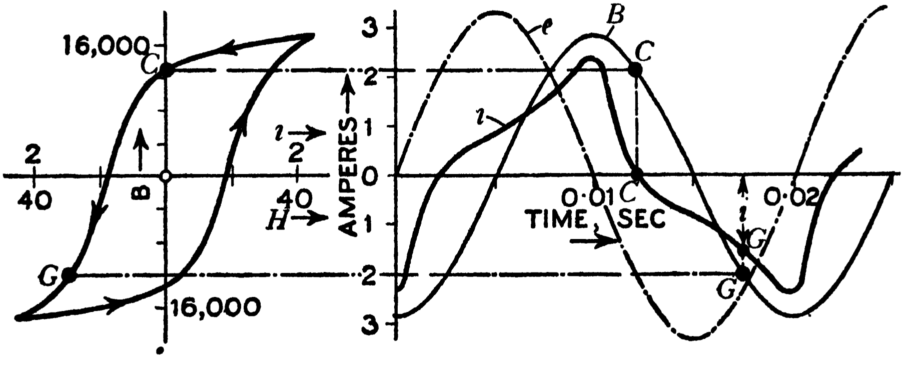


Fig. 7

from the hysteresis loop through the intermediate step of the cosine wave of flux density B will be clear from a reference to the two selected points C and G.

It is to be observed that the applied pressure wave is assumed to be a sine function of the time, and consequently the induced magnetic flux must also be a sine function of the time, since the rate of change of a sine wave of flux gives a sine wave of induced e.m.f.

5.—In Fig. 8 is shown an oscillatory circuit to the terminals of which is applied a d.c. pressure of V volts. Find the logarithmic decrement and the amplitude of the undamped current.

ANSWER.—If $\left(\frac{1}{2}\frac{R}{L}\right)^2$ is less than $\frac{1}{LC}$ the current will be oscillatory as shown in Fig. 8 and is defined by the equation

$$i = V\sqrt{\frac{C}{L}}\left(e^{-\frac{1}{2}\frac{R}{L}t}\right)\sin\left(\frac{1}{\sqrt{LC}}\right)t$$

The frequency f is given by the relationship

$$\omega = 2\pi f = \frac{1}{\sqrt{LC}}$$

and the time of one complete oscillation will be

$$\tau = \frac{1}{f} = 2\pi\sqrt{LC}$$

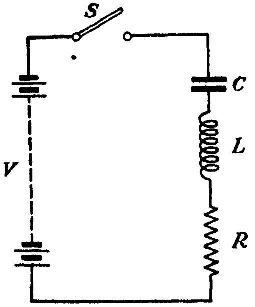


Fig. 8

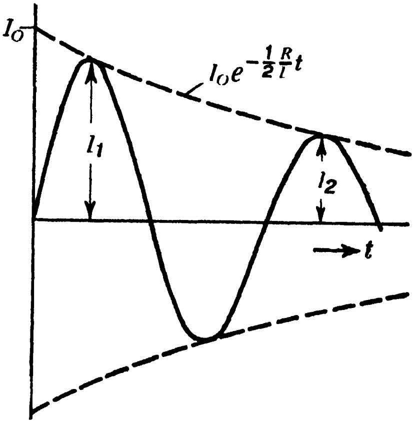


Fig. 9

The first positive peak value will be

$$I_1 = V\sqrt{\frac{C}{L}}\,e^{-\frac{1}{2}\frac{R}{L}\frac{\tau}{4}}$$

and the second positive peak value will be

$$I_2 = V\sqrt{\frac{C}{L}}\,e^{-\frac{1}{2}\frac{R}{L}\frac{5\tau}{4}}$$

The logarithmic decrement will be

$$\triangle = \log_e \frac{I_1}{I_2} = \log_e e^{-\frac{1}{2}\frac{R}{L}\tau}$$

or

$$\triangle = \frac{1}{2}\frac{R}{L}\tau = \frac{1}{2}\frac{R}{L}2\pi\sqrt{LC} = \underline{\pi R\sqrt{\frac{C}{L}}}$$

The undamped peak value will be given by the relationship

$$I_1 = I_0\, e^{-\frac{1}{2}\frac{R}{L}\frac{\tau}{4}}$$

or

$$I_0 = I_1\, e^{\frac{1}{2}\frac{R}{L}\frac{\tau}{4}} \simeq I_1\left(1 + \frac{1}{2}\frac{R}{L}\frac{\tau}{4}\right)$$

that is

$$I_0 \simeq I_1\left(1 + \frac{\pi}{4}R\sqrt{\frac{C}{L}}\right) = I_1\left(1 + \frac{1}{4}\triangle\right)$$

6.—What is meant by the " Q " value of an alternating current circuit ?

Answer.—The " Q " value is the ratio $\frac{\omega L}{R}$ and is a convenient factor for expressing various characteristic relationships of tuned circuits. For example, in a series resonance circuit as shown in Fig. 10 for small values of R the condition of resonance is $\omega L = \frac{1}{\omega C}$ so that

$$I_m = \frac{E_m}{R} : \ (V_l)_m = \omega L\, I_m$$

so that

$$\frac{(V_L)_m}{E_m} = \frac{\omega L}{R} = Q$$

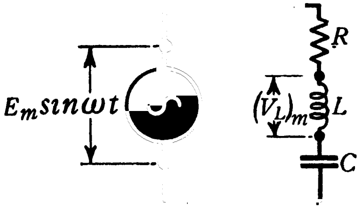


Fig. 10

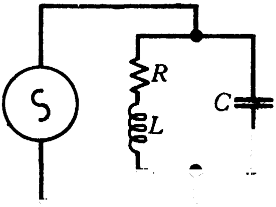


Fig. 11

In the parallel circuit of Fig. 11 the input admittance is

$$\mathfrak{B}_1 = j\omega C + \frac{1}{R + j\omega L} = \frac{j\omega CR - \omega^2 LC + 1}{R + j\omega L}$$

At resonance $\omega^2 LC = 1$ and the input impedance is then

$$\mathfrak{Z}_1 = \frac{I}{\mathfrak{Y}_1} = \frac{R + j\omega L}{j\omega CR}$$

For relatively small values of R:

$$\mathfrak{Z}_1 = \frac{\omega L}{\omega C R} = \omega L \frac{\omega L}{R}$$

that is, $$\mathfrak{Z}_1 = \frac{1}{\omega C} Q = \omega L Q$$

7.—A condenser is connected in parallel with a series arrangement of resistance and inductance. This parallel system is then connected to a source of a.c. supply. Discuss the resonance conditions of the system.

ANSWER.—The circuit connections are shown in Fig. 12. The input impedance is

$$\mathfrak{Z}_1 = \frac{R\dfrac{1}{(\omega C)^2} - j\dfrac{1}{\omega C}\left\{R^2 + (\omega L)^2 - \dfrac{L}{C}\right\}}{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$$

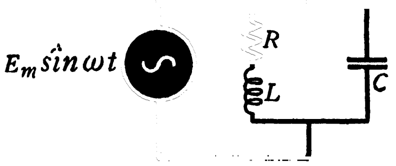


Fig. 12

Taking the following numerical data:

$R = 50\,\Omega$: $C = 2{\cdot}5 \times 10^{-9}\,F$: $L = 10^{-4}$ H: $E_m = 1{,}000$ volts.

Case i: $$\omega = \omega_o = \underline{\frac{1}{\sqrt{LC}}}$$

so that $$\underline{\omega_o} = \sqrt{\frac{10^9 \times 10^4}{2{\cdot}5}} = \underline{2 \times 10^6}$$

and $$\mathfrak{Z}_1 = 800 - j\,200 = 825 e^{-j\,14^\circ}$$

$$\mathfrak{I}_o = \frac{1{,}000}{825} e^{j\,14^\circ} = \underline{1{\cdot}212\, e^{j\,14^\circ}} \text{ (see Fig. 13}a)$$

Capacitance current is $\underline{I_c = 5 \text{ amperes}}$

Case ii: $\omega = \omega_n = \sqrt{\omega_o^2 - \left(\dfrac{1}{2}\dfrac{R}{L}\right)^2}$ that is, the natural frequency of the circuit.

then $$\underline{\omega_n} = \sqrt{4 \times 10^{12} - (0{\cdot}25)^2 \times 10^{12}} \doteq \underline{1{\cdot}983 \times 10^6}$$

$$\mathfrak{Z}_1 = 809\cdot53 - j\,146\cdot1 = 825\cdot6\,e^{-j\,10^\circ\,12'}$$

$$\mathfrak{I}_o = \frac{1,000}{825}\,e^{j\,10^\circ\,12'} = 1\cdot2100 e^{j\,10^\circ\,12'} \text{ (see Fig. 13}b\text{)}$$

Capacitance current is $I_c = \underline{4\cdot95 \text{ amperes.}}$

Case iii: The Current is in Phase with the Supply Pressure.

For this condition the j component of the input impedance must vanish,

that is, $$R^2 + (\omega L)^2 - \frac{L}{C} = 0$$

so that $$\omega^2 = \frac{1}{LC} - \left(\frac{R}{L}\right)^2 :$$

that is, $$\underline{\omega = 1\cdot938 \times 10^6}$$

and $Zi = 820\ \Omega$

$$\underline{\mathfrak{I}_o} = \frac{1,000}{820} = \underline{1.2195} \text{ amperes in phase with the supply pressure.}$$

Fig. 13*c*.)

Capacitance current is $\underline{I_c = 4\cdot87 \text{ amperes.}}$

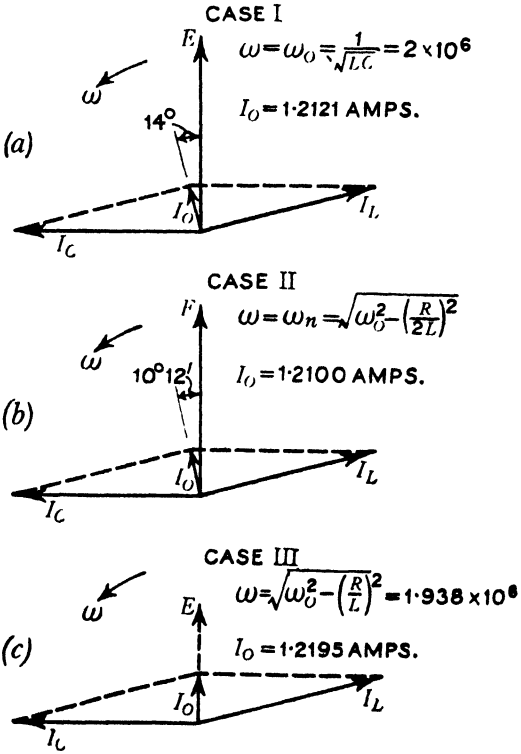

CASE I
E
$\omega = \omega_0 = \frac{1}{\sqrt{LC}} = 2\times10^6$
ω
I_0 = 1·2121 AMPS.
14°
(a)
I_0
I_L
I_C
CASE II
E
$\omega = \omega_n = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$
ω
10°12′
I_0 = 1·2100 AMPS.
(b)
I_0
I_L
I_C
CASE III
ω
E
$\omega = \sqrt{\omega_0^2 - \left(\frac{R}{L}\right)^2} = 1\cdot938\times10^6$
(c)
I_0 = 1·2195 AMPS.
I_0
I_L
I_C

Fig. 13

8.—The two circuits 1 and 2 shown in Fig. 14 are electromagnetically coupled, the coefficient of mutual inductation being M henry, and the coefficients of self-induction respectively L_1 and L_2 henry, and the resistances R_1 and R_2. If it be assumed that there is no magnetic leakage, obtain an expression for the transformation ratio.

ANSWER.—The transformation ratio is defined as the ratio of the induced e.m.f.s in the two circuits (see Fig. 14), viz.,

$$u = \frac{E_1}{E_2}$$

where circuit ① is the primary, i.e., the one connected to the supply and circuit ② is the secondary or consumer's circuit. If ω is the circular frequency of the supply, then

$$E_1 = \omega \, L_1 \, I_1$$

and, by the definition of mutual inductance,

$$E_2 = \omega \, M \, I_1$$

so that the transformation ratio is

$$u = \frac{E_1}{E_2} = \frac{L_1}{M} \quad \text{......................} \quad (1)$$

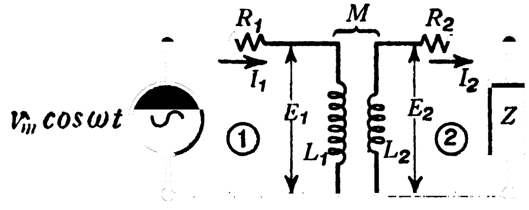


Fig. 14

Similarly, if the circuit ② is connected to the supply, then

$$E_2 = \omega \, L_2 \, I_2 : \; E_1 = \omega \, M \, I_2$$

so that

$$u = \frac{E_2}{E_1} = \frac{M}{L_2} \quad \text{......................} \quad (2)$$

Combining equations (1) and (2) gives

$$M^2 = L_1 \, L_2$$

If there are w_1 turns on the inductance L_1 and w_2 turns in the inductance L_2 then

$$u = \frac{E_1}{E_2} = \frac{L_1}{M} = \frac{L_1}{\frac{w_2}{w_1} L_1} = \frac{w_1}{w_2}$$

that is,

$$\underline{\underline{u = \frac{w_1}{w_2} = \frac{M}{L_2} = \frac{L_1}{M}}}$$

9.—Two coupled circuits shown in the accompanying diagram, the primary circuit of which is connected to an a.c. supply pressure V_m cos ωt. Obtain (i) the two simultaneous equations which define the current in each of the coupled circuits, and (ii) the simple series circuit which is electromagnetically equivalent to the coupled system of circuits.

ANSWER.—The circuit system is shown in Fig. 15.

(*i*)
$$\left.\begin{aligned} V_m \cos \omega t &= L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} + R_1 i_1 \\ O &= L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} + R_2 i_2 \end{aligned}\right\} \begin{aligned} &\ldots\ldots\ldots\ldots\ldots (a) \\ &\ldots\ldots\ldots\ldots\ldots (b) \end{aligned}$$

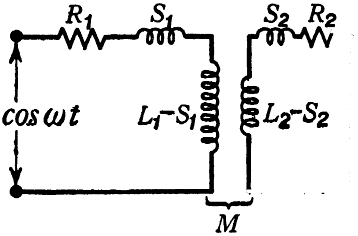


Fig. 15

(*ii*) To solve these equations, assume the solutions to be

$$i_1 = I_{1m}\, e^{j(\omega t - \varphi_1)} : \ i_2 = I_{2m}\, e^{j(\omega t - \varphi_2)}$$

Writing $V_m\, e^{j\omega t}$ for the supply pressure and substituting for i_1 and i_2 in the original simultaneous equations, (*a*) and (*b*) gives

$$V_m\, e^{j\omega t} = L_1 j\, \omega\, I_{1m}\, e^{j(\omega t - \varphi_1)} + M j\, \omega\, I_{2m}\, e^{j(\omega t - \varphi_2)} + R_1 I_{1m}\, e^{j(\omega t - \varphi_1)}$$

$$O = L_2 j\, \omega\, I_{2m}\, e^{j(\omega t - \varphi_2)} + M j\, \omega\, I_{1m}\, e^{j(\omega t - \varphi_1)} + R_2 I_{2m}\, e^{j(\omega t - \varphi_2)}$$

Eliminating $I_{2m}\, e^{j(\omega t - \varphi_2)}$ between these two equations, gives

$$V_m\, e^{j\omega t} = I_{1m}\, e^{j(\omega t - \varphi_1)} \left\{ (R_1 + j\, \omega\, L_1) + j\, \omega\, M \left(\frac{-j\, \omega\, M}{R_2 + j\, \omega\, L_2} \right) \right\}$$

which reduces to

$$V_m\, e^{j\omega t} = I_{1m}\, e^{j(\omega t - \varphi_1)} \{ R_e + j\, \omega\, L_e \} = I_{1m}\, e^{j(\omega t - \varphi_1)}\, \mathfrak{Z}_e$$

where
$$R_e = R_1 + \left(\frac{\omega M}{Z_2} \right)^2 R_2 = R_1 + R_2{}^1$$

$$L_e = L_1 - \left(\frac{\omega M}{Z_2} \right)^2 L_2 = L_1 - L_2{}^1$$

$$\mathfrak{Z}_e = R_e + j\, \omega\, L_e$$

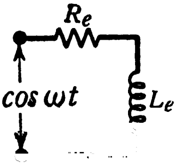


Fig. 16

so that

$R_2{}^1$ is the effective resistance of the secondary circuit reduced to the primary circuit.

L_2^1 is the effective inductance of the secondary circuit reduced to the primary circuit,

and the equivalent simple series circuit is shown in Fig. 16.

If the resistance R_2 is small in comparison with (ωL_2), then $\mathfrak{Z}_2 \stackrel{\wedge}{=} \omega L_2$

and
$$\left(\frac{\omega M}{\mathfrak{Z}_2}\right)^2 \stackrel{\wedge}{=} u^2 = \left(\frac{w_1}{w_2}\right)^2$$

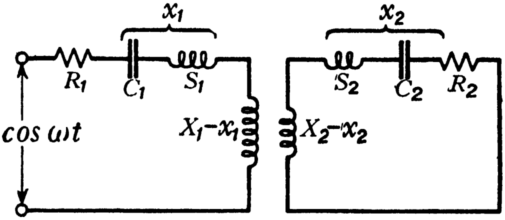


Fig. 17

The same general solution holds for the case in which either, or both, circuits contain capacitances as shown in Fig. 17. If instead of ωL_1 is written X_1 then :

$$X_1 = j\left(\omega L_1 - \frac{1}{\omega C_1}\right) : X_2 = j\left(\omega L_2 - \frac{1}{\omega C_2}\right)$$

$$\mathfrak{Z}_2 = R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)$$

$$X_e = X_1 - \left(\frac{\omega M}{Z_2}\right)^2 X_2$$

10.—A trip coil for a circuit-breaker has an inductance of 0.1 mH and a very small capacitance, e.g., the leading-in conductor from the circuit-breaker, the capacitance of which might be 10^{-4} μF. Find the natural frequency of this coil.

ANSWER.—The natural circular frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0{\cdot}1 \times 10^{-3} \times 10^{-10}}} = 10^7$$

and
$$f_0 = \frac{\omega_0}{2\pi} = 1{\cdot}6 \times 10^6\, H_z.$$

The surge impedance or " oscillation resistance " (see Chapter XV) is

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0{\cdot}1 \times 10^{-3}}{10^{-10}}} = 1{,}000\ \Omega$$

11.—***Find the natural frequency of oscillation of a protective choke coil of inductance L = 5 mH which is connected in front of a cable of capacitance C = 2μF.***

Answer.—The natural circular frequency is

$$\omega_0 = \frac{1}{\sqrt{5 \times 10^{-3} \times 2 \times 10^{-6}}} = 10{,}000$$

and
$$f_0 = \frac{\omega_0}{2\pi} = 1{,}600\ H_z.$$

The surge impedance or " oscillation resistance " is

$$Z_0 = \sqrt{\frac{5 \times 10^{-3}}{2 \times 10^{-6}}} = 50\,\Omega.$$

From these results and those of Example (10) it is seen that the range of natural frequencies and oscillation resistances met with in heavy current electrical engineering is a very wide one.

12.—***Experiments show that when an arc is formed in oil, oil-gas is generated, the volume of which is about 50 c.cm. per kW. sec. at normal temperature and pressure. The relationship between the pressure, volume, and absolute temperature of a gas is*** $\frac{PV}{\Theta}$ = ***constant. If on opening a switch to interrupt*** 0.5×10^6 ***kVA., the generated energy of the arc is 400 kW. sec. and the temperature of the oil-gas generated is 5,000° C., what will be the volume of gas generated if the gas pressure is 25 atmospheres ?***

Answer.—400 kW. sec. will generate 400 × 50 c.cm. = 20 litres of oil-gas at normal temperature (20° C.) and pressure (760 m.). If the temp. is 5,000° C. and the pressure 25 atmospheres, the amount of oil-gas generated will be

$$20\left(\frac{5{,}273}{293}\right)\frac{1}{25} = \underline{15.5 \text{ litres}}.$$

13.—***Explain the action of the " shaded pole " motor—also known as the " Ferraris Disc."***

Answer.—The " shaded pole " type of motor is very widely used for driving the mechanism of relays and for many other purposes. The principle of action was disclosed by Ferraris, hence the name the " Ferraris Disc," and its first important practical application was when Elihu Thomson used it for the operation of electricity supply meters.

In Fig. 18 is shown a laminated iron yoke *Y* excited by a coil *C*. An aluminium disc *D* is so arranged that it can rotate with a segment in the air-gap of an electromagnet. Each pole of the magnet core is provided with a slot so that it can accommodate a seamless copper sleeve.G, the sleeves on the respective poles being directly opposite to each other (only one, however, being visible in Fig. 18). That portion of each pole which is embraced by its sleeve forms the so-called " shaded pole."

When an alternating current I_1 flows in the exciting coil C of the electromagnet, a current I_2' will be induced in the copper sleeves and a current I_2'' will be induced in the aluminium disc D, these currents I_2' and I_2'' being more or less in phase with each other (see Fig. 19). The copper sleeves act like short-circuited secondary coils of a transformer and the magnetic flux in the shaded part of the pole will be reduced to a relatively small amount, whilst in the unshaded portions of the poles the flux can become fully established. The current I_2' in the copper sleeve and I_2'' in the aluminium disc will then produce a mutually attractive force so that a torque is exerted on the disc tending to drive it in the *direction from the unshaded to the shaded portion of the poles* as shown in Fig. 18.

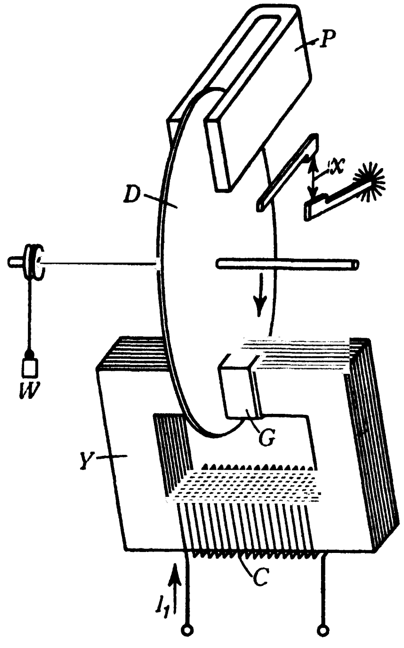


Fig. 18

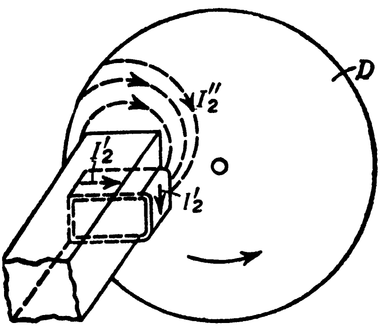


Fig. 19

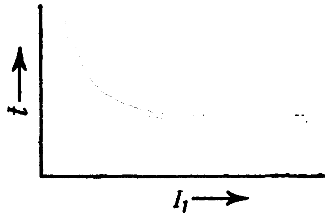


Fig. 20

If a permanent magnet P is arranged as in Fig. 18, the eddy currents to which it will give rise when the disc rotates, will produce a braking torque so that, for a given current I_1 in the exciting coil C a definite steady speed of the disc will be obtained. If a contact is fixed to the disc and another contact is fixed to the framework of the mechanism, these two contacts can be used as the terminals for the excitation of an electromagnetic relay which, for example, actuates the opening of the circuit breaker. Then if x is the distance between the fixed and moving contacts, the time-lag which is obtained before the relay becomes energised will be $t = \frac{x}{v}$ where v is the speed of the moving contact.

In order to obtain the desired shape for the characteristic time-lag curve, as shown in Fig. 20 for example, a constant opposing torque is applied to the disc, and one method for doing this is shown in Fig. 18 in which a mass W is suspended by a cord which is wound on a pulley fixed to the shaft of the disc. By means of the combined action of the opposing torque due to this weight W and the braking torque due to the magnet P, the desired shape of the time-lag characteristic curve can be obtained such as that of Fig. 20.

14.—Show how the short-circuit impedance of a transformer may be measured by means of a Bridge test.

ANSWER.—The short-circuit impedance of a transformer is a strictly constant quantity and independent of the current. In consequence of this, the impedance may be measured by a Bridge method as shown in Fig. 21. At balance :

$$i_1 (R + j X) = i_2 p : i_1 q = i_2 (r - j x_c)$$

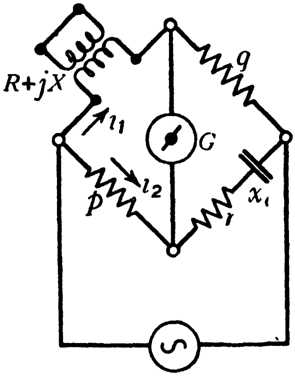


Fig. 21

so that $$\frac{R + j X}{q} = \frac{p}{r - j x_c} : X r = R x_c : Rr + X x_c = pq$$

and $$R = \frac{p q}{r^2 + x_c^2} r : X = \frac{p q}{r^2 + x_c^2} x_c$$

When the impedance $R + j X$ is very small, the contact resistances may introduce errors and it is then desirable to use the Thomson Double Bridge (see page 311, "Principles of Electrical Engineering.")

The advantages of using a Bridge method is its simplicity as well as the fact that, in this way, large power requirements are eliminated. This method can be used for testing transformers of capacity 40 mVA by means of a source of only 40 VA. (See Brown Boveri Bulletin, 1943.)

15.—Compare the characteristic performance of a "polarised" and a "non-polarised" relay.

ANSWER.—An outline sketch of a magnetic relay is shown in Fig. 22 in which the armature is controlled by a spring and the range of movement is limited by two stops. This is a non-polarised or "neutral" relay and the force of attraction on the armature is defined by the expression:

$$F = kB^2$$

If the induction B in the air-gap increases by the small amount $\triangle B$ due to an impulse current signal, the force will be,

$$F + \triangle F = k\,(B + \triangle B)^2 = k\,\{B^2 + 2B(\triangle B) + (\triangle B)^2\}$$

and if $(\triangle B)^2$ is negligibly small then

$$\triangle F = 2kB(\triangle B)$$

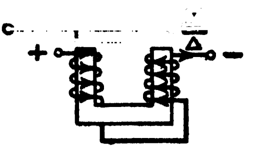


Fig. 22

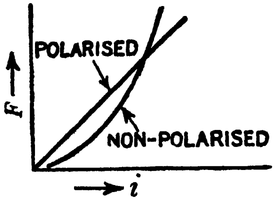


Fig. 23

If the magnet is polarised by means of a permanent magnet (as for example, in the telephone receiver), the effect of the change $\triangle B$ due to the signal current is proportional to the magnitude of the induction density B which is produced by the permanent magnet. For a given operating magneto motive force the polarised relay is more effective for small values of the m.m.f. and the non-polarised relay is the more effective for large values of the m.m.f. as is shown in Fig. 23.

16.—*Compare the action of a magnetic relay operated by alternating current with that of a relay operated by direct current.*

Answer.—When a magnetic relay is magnetised by single phase alternating current, viz., a current defined by the expression $i = I_m \sin \omega t$, the force of attraction on the armature for the unsaturated range of magnetisation will be,

$$f = ki^2 = kI_m^2 \sin^2 \omega t = F_m \sin^2 \omega t$$

that is, $f = \frac{1}{2} F_m\,(1 - \cos 2\omega t)$

from which it is seen that the force alternates with double the frequency of the supply current (see also Example 3 of this Test Paper) as is shown by Fig. 24. The force becomes zero twice during each cycle of the alternating current and consequently a relay operated in this way would tend to chatter. If however a two-phase current were used as defined by the expressions

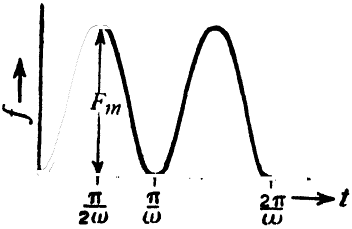


Fig. 24

$$i_1 = I_m \sin \omega t : i_2 = I_m \cos \omega t$$

the force of attraction would then be,

$$f = F_m \sin^2 \omega t + F_m \cos^2 \omega t = F_m$$

and would therefore be constant at every moment just as is the case with d.c. excitation.

A two-phase current may be obtained from a single-phase source of supply in two different ways as follows :

(i). An electromagnet system comprises two circuits in parallel as shown in Fig. 25, each containing a coil which energizes the magnets and the magnetic fluxes due to these two exciting coils are superimposed in a common armature of the relay. The vector diagram of Fig. 26 will be seen to be in accordance with the following relationships, viz. :

The applied pressure vector is

$$\mathfrak{V} = \mathfrak{J}_1 (R + j\omega L) = \mathfrak{J}_2 \left\{ R + j\left(\omega L - \frac{1}{\omega C}\right) \right\}$$

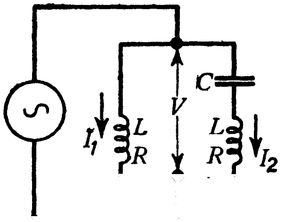


Fig. 25

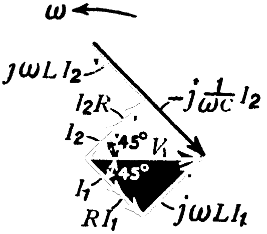


Fig. 26

it being observed that the values of the resistances R and of the inductances L are assumed to be the same for both coils. In order that the two current vectors $\mathfrak{J}_1$ and $\mathfrak{J}_2$ shall be of the same magnitude and shall have a phase displacement of 90° then

$$\mathfrak{J}_2 = j\mathfrak{J}_1$$

so that (Fig. 26) $\mathfrak{V} = \mathfrak{J}_1 (R + j\omega L) = j\mathfrak{J}_1 \left\{ R + j\left(\omega L - \frac{1}{\omega C}\right) \right\}$

that is $$R + j\omega L = j \left\{ R + j\left(\omega L - \frac{1}{\omega C}\right) \right\}$$

or $$R + j\omega L = jR - \omega L + \frac{1}{\omega C}$$

After equating the real quantities and the j quantities on both sides of this equation it is found that

$$R = \frac{1}{\omega C} - \omega L : R = \omega L$$

so that $$R = \frac{1}{2\omega C} : C = \frac{1}{2\omega^2 L}$$

This result means that there must be a condition of resonance between $\mathfrak{B}$ and $\mathfrak{J}$ that is, the reactance $2\omega L$ and the reactance $\frac{1}{\omega C}$ must be equal, so that

$$2\,\omega^2 LC = 1.$$

(ii). The requisite phase displacement between the two magnetic fluxes can be obtained by means of a " shaded pole " electromagnetic system such as is shown in Fig. 27 where S is a seamless copper sleeve, which acts as a short-circuited transformer secondary winding.

The exciting coil *C* produces the flux 2Φ round the main magnetic circuit, and this divides equally between the two prongs of the fork as shown in Fig. 27, on one of which prongs the seamless copper tube S is placed. The induced current in the copper sleeve produces a flux Φ_S, which passes round the prongs and through the air-gap so that in one prong the flux $\Phi + \Phi_S$ will be produced and in the other prong the flux $\Phi - \Phi_S$. The vector relationships for the general case are shown in Fig. 28, and for the special case in which $\Phi + \Phi_S = \Phi - \Phi_S$ in Fig. 29, in which it is seen that these two fluxes are displaced by 90° relatively to one another. The Bell Telephone Company uses this system with an a.c. excitation at 50 frequency.

The time required for the relay to operate depends upon the electromagnetic inertia and the mass inertia. In the system (i) the capacitance *C* (Fig. 25) is chosen as small as possible and the inductance as large as possible. The magnitude chosen for *C* determines the impedance of the relay, and this is selected to suit the particular purpose for which the relay is required.

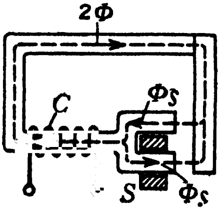


Fig. 27

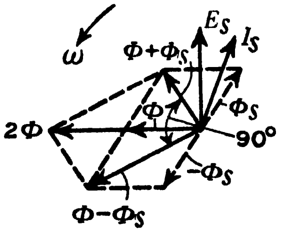


Fig. 28

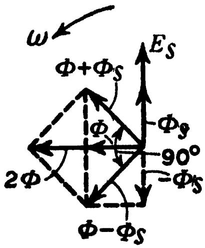


Fig. 29

Chapter X

OSCILLATING SYSTEMS

1.—***Two circuits each comprising a series arrangement of capacitance, inductance, and resistance are coupled by mutual induction. Find the simple series circuit which is electromagnetically equivalent to this coupled circuit.***

ANSWER.—The simultaneous equations in this case are (see Fig. 1)

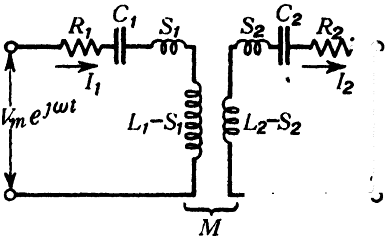


Fig. 1

$$\left.\begin{aligned} V_m e^{j\omega t} &= L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1\,dt \qquad \text{(a)} \\ O &= L_2\frac{di_2}{dt} + M\frac{di_1}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt \qquad \text{(b)} \end{aligned}\right\}$$

Differentiating each of these with respect to t gives

$$\left.\begin{aligned} j\omega V_m e^{j\omega t} &= L_1\frac{d^2 i_1}{dt^2} + M\frac{d^2 i_2}{dt^2} + R_1\frac{di_1}{dt} + \frac{i_1}{C_1} \\ O &= L_2\frac{d^2 i_2}{dt^2} + M\frac{d^2 i_1}{dt^2} + R_2\frac{di_2}{dt} + \frac{i_2}{C_2} \end{aligned}\right\}$$

Assuming the solutions of these equations to be:

$$i_1 = I_{1m}\, e^{j(\omega t - \varphi_1)} \; : \; i_2 = I_{2m}\, e^{j(\omega t - \varphi_2)}$$

then:

$$\left\{\begin{aligned} j\omega V_m e^{j\omega t} &= (j\omega)^2 L_1 I_{1m} e^{j(\omega t - \varphi_1)} + (j\omega)^2 M I_{2m} e^{j(\omega t - \varphi_2)} \\ &\qquad + R_1 j\omega I_{1m} e^{j(\omega t - \varphi_1)} + \frac{1}{C_1} I_{1m} e^{j(\omega t - \varphi_1)} \\ O &= (j\omega)^2 L_2 I_{2m} e^{j(\omega t - \varphi_2)} + (j\omega)^2 M I_{1m} e^{j(\omega t - \varphi_1)} \\ &\qquad + R_2 j\omega I_{2m} e^{j(\omega t - \varphi_2)} + \frac{1}{C_2} I_{2m} e^{j(\omega t - \varphi_2)} \end{aligned}\right.$$

Eliminating I_{2m} between these two equations gives:

$$\mathfrak{V}_m e^{j\omega t} = \frac{I_{1m}}{\omega} e^{j(\omega t - \varphi_1)} \left\{ [j\omega^2 L_1 - j\frac{1}{C_1} + \omega R_1] + j\left(\frac{\omega^2 (\omega M)^2}{-\left(L_2\omega^2 - \frac{1}{C_2}\right) + j\omega R_2} \right) \right\}$$

that is,

$$\mathfrak{V}_m e^{j\omega t} = I_{1m} e^{j(\omega t - \varphi_1)} \left\{ (R_1 + jX_1) + j\left(\frac{(\omega M)^2}{-X_2 + jR_2} \right) \right\}$$

$$= I_{1m} e^{j(\omega t - \varphi_1)} \{ \mathfrak{Z}_e \}$$

where

$$\mathfrak{Z}_e = R_e + jX_e = \left\{ R_1 + \left(\frac{\omega M}{Z_2}\right)^2 R_2 \right\} + j\left\{ X_1 - \left(\frac{\omega M}{Z_2}\right)^2 X_2 \right\}$$

that is,

$$\boxed{\mathfrak{Z}_e = \left\{ R_1 + \left(\frac{X_m}{Z_2}\right)^2 R_2 \right\} + j\left\{ X_1 - \left(\frac{X_m}{Z_2}\right)^2 X_2 \right\}}$$

where $X_m = \omega M$ and is the mutual reactance

$\mathfrak{Z}_2 = R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)$ and is the secondary impedance.

R_e is the equivalent primary resistance,
X_e is the equivalent primary reactance.

so that
$$\left.\begin{aligned} R_e &= R_1 + \left(\frac{X_m}{Z_2}\right)^2 R_2 \\ X_e &= X_1 - \left(\frac{X_m}{Z_2}\right)^2 X_2 \end{aligned}\right\}$$

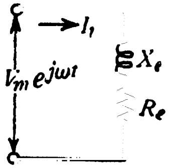


Fig. 2

and the coupled oscillatory system of Fig. 1 can consequently be represented by the simple series circuit of Fig. 2. If $C_1 = C_2 = \infty$ these expressions for R_e and X_e respectively become identical with those which were obtained in the solution for Example 9, Chapter IX.

Numerical Example

$L_1 = 10^{-4}$ H : $C_1 = 5 \times 10^{-9}$ F : $R_1 = 10\,\Omega$
$L_2 = 1{\cdot}5 \times 10^{-4}$ H : $C_2 = 6 \times 10^{-9}$ F : $R_2 = 12\,\Omega$
$M = 0{\cdot}8 \times 10^{-4}$ H : $\omega = 1{\cdot}5 \times 10^6$

$$X_1 = \omega L_1 - \frac{1}{\omega C_1} = 16{\cdot}7\,\Omega : X_2 = \omega L_2 - \frac{1}{\omega C_2} = 113{\cdot}9\,\Omega$$

$$X_m = \omega M = 120\,\Omega : Z_2 = \sqrt{113{\cdot}9^2 + 12^2} = 114{\cdot}3\,\Omega$$

so that

$$\left.\begin{cases} R_e = 10 + \left(\dfrac{120}{114{\cdot}3}\right)^2 12 = 24{\cdot}4\ \Omega \\ X_e = 16{\cdot}7 - \left(\dfrac{120}{114.3}\right)^2 113{\cdot}9 = -\ 108\ \Omega \end{cases}\right\} \mathfrak{Z}_e = (24{\cdot}4 - j\,108)\ \Omega.$$

If the same numerical data are taken as in the foregoing, except that M is now reduced to $0{\cdot}1 \times 10^{-4}\ H$ so that

$$X_m = \omega M = 15\ \Omega,$$

then

$$\left.\begin{matrix} R_e = 10{\cdot}206\ \Omega \\ X_e = 14{\cdot}75\ \Omega \end{matrix}\right\} : \mathfrak{Z}_e = (10{\cdot}206 + j\ 14{\cdot}75)\ \Omega.$$

2.—Find the magnitudes of the primary current and of the secondary current of the mutually coupled circuits of Example 1.

ANSWER.—The secondary current is

$$I_{2m}\, e^{j(\omega t - \varphi_2)} = \left(\frac{\omega^2 M}{-\ L_2\omega^2 + \dfrac{1}{C_2} + j\omega R_2}\right) I_{1m}\, e^{j(\omega t - \varphi_1)}$$

$$= -\left(\frac{\omega M}{L_2\omega - \dfrac{1}{\omega C_2} - jR_2}\right) I_{1m}\, e^{j(\omega t - \varphi_1)}$$

hence,

$$\frac{I_{2m}\, e^{j(\omega t - \varphi_2)}}{I_{1m}\, e^{j(\omega t - \varphi_1)}} = -\ \frac{\omega M\,(X_2 + jR_2)}{Z_2^{\,2}}$$

and the ratio of the current magnitudes is

$$\underline{\frac{I_2}{I_1} = \frac{\omega M}{Z_2}} : \quad \underline{I_2 = \frac{\omega M}{Z_2}\,\frac{V}{\mathfrak{Z}_e}}$$

where $\mathfrak{Z}_e = \sqrt{R_e^2 + X_e^2}$ and is the magnitude of the equivalent primary impedance as defined in Example 1. Writing these results in full, therefore and *assuming a condition of resonance so that* $X_2 = 0$:

$$\frac{I_2}{I_1} = \frac{\omega M}{\sqrt{R_2^2 + X_2^2}} :$$

$$I_2 = \frac{\omega M V}{\sqrt{\left\{R_1 + \left(\dfrac{X_m}{Z_2}\right)^2 R_2\right\}^2 + \left\{X_1 - \left(\dfrac{X_m}{Z_2}\right)^2 X_2\right\}^2}\sqrt{X_2^2 + R_2^2}}$$

If the circuits are separately tuned to the same frequency, then, $X_1 = 0 = X_2$ and,

$$I_1 = \frac{V R_2}{R_1 R_2 + (\omega M)^2}$$

3.—The two coupled circuits of Example 1 are separately tuned to the same resonance frequency, the individual constants of the circuits being as follows:

$$\left.\begin{array}{l} L_1 = 10^{-4}\ H.,\ C_1 = 5 \times 10^{-9}\ F.:\ R_1 = 10\ \Omega \\ L_2 = 1.5 \times 10^{-4}\ H.,\ C_2 = 3.3 \times 10^{-9}\ F.:\ R_2 = 12\ \Omega \end{array}\right\} \omega_0 = 1.4 \times 10^{6}.$$

Find the current in the secondary circuit and obtain the value of the coupling factor for which this current reaches a maximum value.

Answer.—From the results obtained in Example 2 for the primary and secondary currents, it will be seen that when the resonance frequency of the coupled system is near the frequency to which each of the circuits has been separately tuned, then the resistance of each circuit is the dominant quantity in the impedance. It will also be observed that these conditions imply *weak coupling*, that is, a low value for the coupling factor $K = \frac{M}{\sqrt{L_1 L_2}}$.

Assuming then that X_1 and X_2 are each very small, the magnitude of the secondary current will be given by

$$I_2 = \frac{\omega M}{Z_2}\,\frac{V}{Z_e} \simeq \frac{\omega M}{R_2}\,\frac{V}{R_1 + \frac{(\omega M)^2}{R_2}}$$

$$I_1 = \frac{V R_2}{R_1 R_2 + (\omega M)^2}$$

that is, $$I_2 = \frac{\omega M V}{R_1 R_2 + (\omega M)^2} : \frac{I_1}{I_2} = \frac{R_2}{\omega M}$$

For maximum secondary current,

$$\frac{dI_2}{d\omega} = 0 : \text{that is, } \frac{MV}{R_1 R_2 + (\omega M)^2} - \frac{\omega M V\, 2\omega M^2}{[R_1 R_2 + (\omega M)^2]} = 0$$

or $$R_1 R_2 + (\omega M)^2 = 2(\omega M)^2$$

that is $$\boxed{(\omega M)^2 = R_1 R_2}$$

In the present example the given numerical data lead to $(\omega M)^2 = 120$: $\omega M = 10{\cdot}95$, so that the critical coupling factor is obtained from the relationships

$$K^2 = \frac{M^2}{L_1 L_2} = \left(\frac{10{\cdot}95}{\omega_o}\right)^2 \frac{10^4 \times 10^4}{1{\cdot}5} \text{ and since } \omega_o = 1{\cdot}4 \times 10^6$$

$$\underline{K = 0{\cdot}063} : \underline{M = 0{\cdot}0775 \times 10^{-4}}\ \text{H.}$$

Plotting the input admittance (or the primary current) as a function of the frequency and taking $V = 1{,}000$ volts, then,

$$I_1 = \frac{V}{\sqrt{\left[R_1 + \frac{(M\omega)^2}{R_2}\right]^2}} = \frac{VR_2}{R_1 R_2 + (\omega M)^2} = \frac{1000\, R_2}{120 + \left(\frac{0{\cdot}0775}{10^4}\right)^2 \omega^2}$$

and, putting $\omega = \omega_0 = 1{\cdot}41 \times 10^6$ gives $I_1 = 5{\cdot}0 : I_2 = 4{\cdot}57$ amperes.

The conditions for the critical value of the coupling factor may also be expressed as follows:

$$R_1R_2 = (\omega M)^2 : M = K\sqrt{L_1L_2}$$

therefore $$R_1R_2 = K^2(\omega L_1)(\omega L_2)$$

or $$\frac{1}{K^2} = \frac{\omega L_1}{R_1} \times \frac{\omega L_2}{R_2} = Q_1 \times Q_2$$

so that the critical coupling factor is given by the relationship

$$K = \frac{1}{\sqrt{Q_1 \times Q_2}}$$

(See also Example 6 : Chapter IX.)

4.—Two coupled circuits, each tuned separately to the same resonance frequency, have the same numerical value as those given in Example 3:

(i) Find the coupling factor which will give the maximum current in the secondary winding and plot the current-frequency relationships for each of the two circuits.

(ii) Taking a coupling factor equal to one-fifth of the critical value, plot the current-frequency relationships for each of the two circuits.

(iii) Taking a coupling factor equal to five times the critical value, plot the current frequency relationships for each of the two circuits.

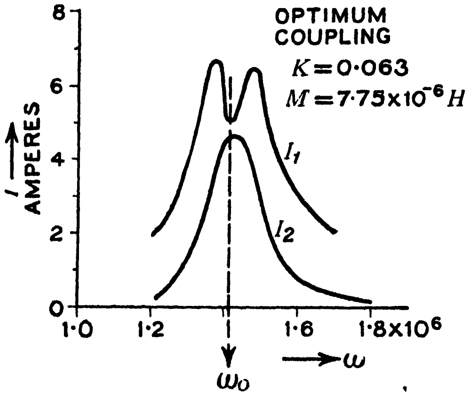


Fig. 3

ANSWER.—(i). The "critical" or "optimum" coupling factor is given by the relationship—

$$K^2 = \frac{M^2}{L_1L_2} \qquad \text{where } (\omega_0 M)^2 = R_1R_2$$

so that
$$K^2 = \frac{R_1R_2}{\omega_0^2 L_1 L_2}$$

or
$$K = \sqrt{\frac{120 \times 10^8}{2 \times 10^{12} \times 1{\cdot}5}} \quad = \underline{0{\cdot}063}$$

and
$$M = \sqrt{\frac{R_1R_2}{\omega_0^2}} = \underline{7{\cdot}75 \times 10^{-6}\,\text{H}.}$$

Finding the respective values for I_1 and I_2 as functions of the supply frequency as in the preceding Example 3, taking for example $V = 100$ volts, the curves shown in Fig. 3 are obtained.

A typical example of such calculations is as follows for the frequency $\omega = 0{\cdot}92\omega_0 = 1{\cdot}3 \times 10^6 : \omega^2 = 1{\cdot}69 \times 10^{12}\ \omega M = 10{\cdot}1$

$$X_1 = 1{\cdot}3 \times 10^2 - \frac{10^9}{1{\cdot}3 \times 10^6 \times 5} = -24\Omega$$

$$X_2 = 1{\cdot}3 \times 1{\cdot}5 \times 10^2 - \frac{10^9}{1{\cdot}3 \times 10^6 \times 3{\cdot}3} = -37\Omega$$

$$Z_2 = \sqrt{37^2 + 12^2} = 38{\cdot}7\Omega$$

$$Z_e = \sqrt{\left\{10 + \left(\frac{10{\cdot}1}{38{\cdot}7}\right)^2 12\right\}^2 + \left[-24 + \left(\frac{10{\cdot}1}{38{\cdot}7}\right)^2 37\right]^2}$$

that is,

$$Z_e = 25\Omega :$$

$$Y_e = \frac{1}{25} = 0{\cdot}040 \text{ siemens} : I_1 = VY_e = 4{\cdot}0 \text{ amperes}$$

$$I_2 = \left(\frac{\omega M}{Z_2}\right) I_1 = 1{\cdot}04 \text{ amperes}$$

as given in Fig. 3.

(ii). Coupling Factor = one-fifth of the optimum value = 0·0126 : $M = 1{\cdot}54 \times 10^{-6}$ H.

Typical calculation : $\omega = 1{\cdot}39 \times 10^6 : \omega M = 2{\cdot}14 :$

$$X_1 = 1{\cdot}39 \times 10^2 - \frac{10^3}{1{\cdot}39 \times 5} = 139 - 144 = -5\Omega$$

$$X_2 = -8\Omega : Z_2 = \sqrt{8^2 + 12^2} = 14{\cdot}4\Omega$$

$$Z_e = \sqrt{\left\{10 + \left(\frac{2{\cdot}14}{14{\cdot}4}\right)^2 12\right\}^2 + \left\{-5 + \left(\frac{2{\cdot}14}{14{\cdot}4}\right)^2 8\right\}^2} = 11{\cdot}38\Omega$$

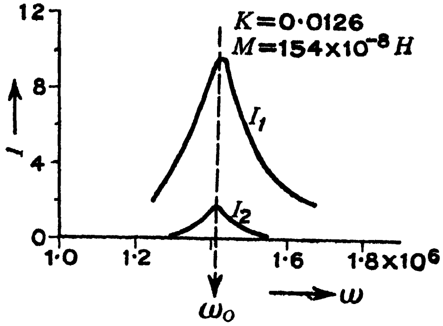


Fig. 4

$$Y_e = 0{\cdot}088 \text{ siemens} : I_1 = VY_e = 8{\cdot}8 \text{ amperes}$$

$$I_2 = \frac{\omega M}{Z_2} I_1 = 1{\cdot}30 \text{ amperes}$$

and in this way the graphs shown in Fig. 4 have been obtained.

(iii). Coupling Factor = 5 times the optimum value, that is,

$$K = 5 \times 0{\cdot}063 = 0{\cdot}315$$

so that $$M = 0{\cdot}315\sqrt{L_1 L_2} = 0{\cdot}386 \times 10^{-4}\ H.$$

A typical calculation in this case is as follows :—

$$\omega = 1{\cdot}7 \times 10^6 : \omega M = 66 : X_1 = 170 - \frac{10^3}{8{\cdot}5} = 52{\cdot}2\ \Omega$$

$$X_2 = 255 - \frac{10^3}{5{\cdot}62} = 78\ \Omega : Z_2 = 78{\cdot}6\ \Omega$$

$$\left(\frac{\omega M}{Z_2}\right)^2 = 0{\cdot}7 :$$

$$Z_e = \sqrt{(10 + 8{\cdot}4)^2 + (52{\cdot}2 - 54{\cdot}5)^2} = 21\ \Omega$$

$$I_1 = \frac{V}{Ze} = 4{\cdot}8 \text{ amperes} : I_2 = \left(\frac{\omega M}{Z_2}\right) I_1 = 4{\cdot}05 \text{ amperes}$$

and in this way the graphs of Fig. 5 have been derived.

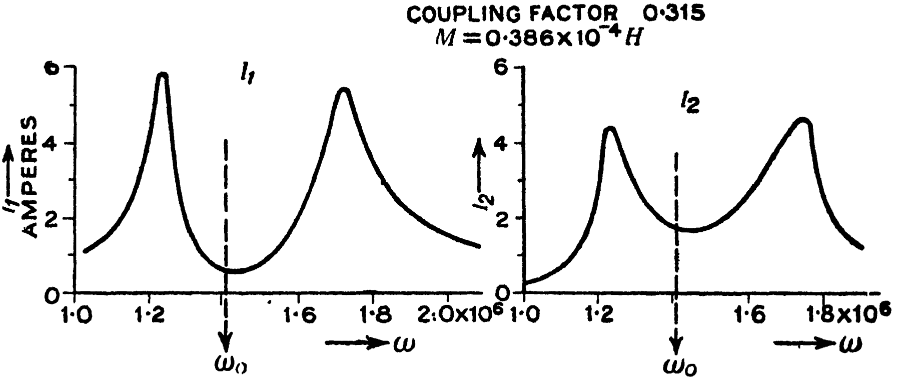


Fig. 5

5.—Find the resonance frequency of two circuits which are coupled by mutual induction, each of the circuits being separately tuned to the same resonance frequency and the resistance of each circuit being relatively small.

Answer.—When the resistance of each circuit is relatively small the input impedance becomes—

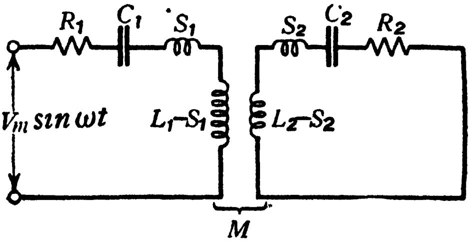


Fig. 6

$$\mathfrak{Z}_i = j\left\{X_1 - \left(\frac{X_m}{Z_2}\right)^2 X_2\right\} \text{ ohms}$$

that is,
$$\mathfrak{Z}_i = j\left\{\left(\omega L_1 - \frac{1}{\omega C_1}\right) - \frac{(\omega M)^2}{\omega L_2 - \dfrac{1}{\omega C_2}}\right\} \text{ ohms}$$

and at resonance frequency this impedance will become zero, that is,

$$\omega^2 L_1 L_2 - \frac{L_1}{C_2} - \frac{L_2}{C_1} + \frac{1}{\omega^2 C_1 C_2} = (\omega M)^2$$

or $$\omega^2 - \frac{1}{L_2 C_2} - \frac{1}{L_1 C_1} + \frac{1}{\omega^2 L_1 C_1 L_2 C_2} = \frac{\omega^2 M^2}{L_1 L_2} = \omega^2 K^2$$

where $K = \frac{M}{\sqrt{L_1 L_2}}$ and is the coupling factor. Since the two circuits are separately tuned to the same resonance frequency ω_0 that is

$$\omega_0{}^2 = \frac{1}{L_1 C_1} = \frac{1}{L_2 C_2}$$

so that $$\omega^2 - 2\omega_0{}^2 + \frac{\omega_0{}^4}{\omega^2} = \omega^2 K^2$$

or $$K^2 = 1 - 2\left(\frac{\omega_0}{\omega}\right)^2 + \left(\frac{\omega_0}{\omega}\right)^4$$

that is $$K = \pm\left\{1 - \left(\frac{\omega_0}{\omega}\right)^2\right\}$$

or $$\left(\frac{\omega_0}{\omega}\right)^2 = 1 \pm K$$

and $$\boxed{\omega = \left.\begin{matrix}\omega_1 \\ \omega_2\end{matrix}\right\} = \frac{1.}{\sqrt{1 \pm K|}}\,\omega_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

It is seen therefore that when two oscillatory circuits are coupled together in the manner assumed, there are two resonant frequencies, the values of which will be determined by the coupling factor K. If this factor has the value 1, that is, the coupling is very " tight, "there will only be one resonant frequency which can be actually realized, viz. :—

$$\omega_1 = \frac{\omega_0}{\sqrt{2}}$$

since the other resonant frequency will be $\omega_2 = \infty$.

(*i*) *When K is small, that is, the coupling is " weak."* In this case the foregoing expression for the resonance frequencies is not generally applicable since for weak coupling the two resonance frequencies approach the value ω_0 to which each circuit is separately tuned. For frequencies which are not far removed from ω_0 however, X_1 and X_2 are both very small and consequently the dominant factor of the input impedance $\mathfrak{Z}_i$ is the real component R_e (Example 4); that is, for very weak coupling—

$$\mathfrak{Z}_i \simeq R_1 + \left(\frac{X_m}{R_2}\right)^2 R_2 = R_1 + \frac{(\omega M)^2}{R_2}$$

The magnitude of the primary current will then be

$$I_1 \simeq \frac{V}{Z_i} = \frac{VR_2}{R_1 R_2 + (\omega M)^2}$$

where V is the magnitude of the p.d. of frequency ω which is applied to the input terminals.

Similarly, the current in the secondary circuit will be

$$I_2 = \frac{\omega M}{Z_2} I_1 \simeq \frac{\omega M}{R_2} I_1 = \frac{\omega MV}{R_1 R_2 + (\omega M)^2}$$

To obtain the maximum value of I_2 it is only necessary to differentiate this expression with respect to ω and to equate the result to zero. That is, for maximum secondary current $I_2]_{\max}$.

$$\frac{d}{d\omega}\left[\frac{\omega MV}{R_1 R_2 + (\omega M)^2}\right] = 0$$

that is $$\boxed{(\omega M)^2 = R_1 R_2}$$

is the condition for maximum secondary current at resonance, that is, the condition for *transfer of maximum power from the primary to the secondary circuit*.

The "critical coupling" to achieve this result is given by

$$K^2 = \frac{M^2}{L_1 L_2} = \frac{R_1 R_2}{\omega^2} \frac{1}{L_1 L_2}$$

or $$\boxed{K = \sqrt{\left(\frac{R_1}{\omega L_1}\right)\left(\frac{R_2}{\omega L_2}\right)} = \sqrt{\frac{1}{Q_1 \times Q_2}}}$$

as has been obtained already in Example 3.

(*ii*) *When the coupling factor is large*, that is, for "tight" coupling, the expression for the two resonance frequencies, viz., $\omega = \frac{1}{\sqrt{1 \pm K}}$ shows that these are well removed from the frequency ω_0 and consequently the resistance of each circuit will be small as compared with the reactances X_1 and X_2 respectively, so that the foregoing expression for the two coupled resonance frequencies can be applied.

6.—Find the input impedance and the resonance frequencies of the compound circuit shown in the accompanying diagram when $C_1 = C_2$ and $L_1 = L_2$.

Answer.—The system comprises the components ① and ②, coupled by the inductance M, and the coupling coefficient is defined as :—

$$K = \frac{M}{M + L_1}$$

The admittances of the respective components of the system are then

$$\mathfrak{Y}_1 = -j\frac{1}{\left(\omega L_1 - \frac{1}{\omega C_1}\right)}$$

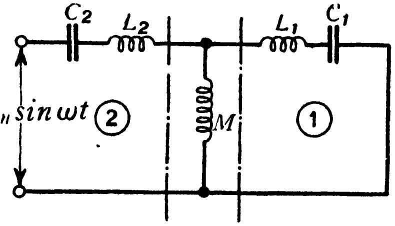


Fig. 7

$$\mathfrak{Y}_M = -j\frac{1}{\omega M}$$

$$\mathfrak{Y}_2 = -j\frac{1}{\omega L_1 - \frac{1}{\omega C_1}}$$

Then $$\mathfrak{Y}_1 + \mathfrak{Y}_M = -j\left\{\frac{1}{\left(\omega L_1 - \frac{1}{\omega C_1}\right)} + \frac{1}{\omega M}\right\}$$

$$\mathfrak{Z}_{1+M} = \frac{1}{\mathfrak{Y}_1 + \mathfrak{Y}_M} = j\left\{\frac{\omega M\,(\omega^2 L_1 C_1 - 1)}{\omega^2\,(MC_1 + L_1 C_1) - 1}\right\}$$

The input impedance is then

$$\mathfrak{Z}_i = \mathfrak{Z}_2 + \mathfrak{Z}_{1+M} = j\left(\omega L_1 - \frac{1}{\omega C_1}\right) + j\left\{\frac{\omega M\,(\omega^2 L_1 C_1 - 1)}{\omega^2\,(MC_1 + L_1 C_1) - 1}\right\}$$

that is, $$\boxed{\mathfrak{Z}_i = j\left\{\frac{(\omega^2L_1C_1 - 1)\,[\omega^2C_1\,(M + L_1) - 1 + \omega^2MC_1]}{\omega C_1\,[\omega^2C_1\,(M + L_1) - 1]}\right\}}$$

When $[\omega^2C_1(M + L_1) - 1] = 0$, then $\mathfrak{Z}_1 = \infty$ (i.e., "parallel" resonance) and the frequency corresponding to this condition is seen to be the natural frequency of the parallel system $M : 1$ of Fig. 7, when the component 2 is removed, viz.:—

$$\boxed{\omega_o^2 = \frac{1}{C_1(M + L_1)}}$$

The input impedance Z_i becomes zero (i.e. "series resonance") for two different conditions, viz.:

(i) $\omega^2L_1C_1 = 1$, and (ii) $\omega^2C_1\,(2M+L_1) = 1$

Case (i).

$$\omega^2 = \frac{1}{L_1C_1} = \omega_o^2\,\frac{M}{KL_1} = \omega_o^2\,\frac{M + L_1}{L_1}$$

or $$\underline{\omega^2 = \omega_o^2\,\frac{1}{1-K}}$$

Case (ii).

$$\omega^2 = \omega_0^2\,\frac{M + L_1}{2M + L_1} = \omega_0^2\,\frac{M}{K}\left[\frac{1}{M + \frac{M}{K}}\right]$$

that is $$\underline{\omega^2 = \omega_0^2\,\frac{1}{1 + K}}$$

The resonance frequencies defined in the foregoing Cases (i) and (ii) respectively can therefore be expressed as follows

$$\boxed{\left.\begin{matrix}\omega_i \\ \omega_{ii}\end{matrix}\right\} = \omega_0\,\frac{1}{\sqrt{1 \pm K}}}$$

Thus, if $C_1 = 2{\cdot}5 \times 10^{-9}\,F : L_1 = 10^{-5}\,H : K = 0{\cdot}333$

$$M = L_1\,\frac{K}{1 - K} = 0{\cdot}5 \times 10^{-5}\,H.\text{: so that,}$$

$$\omega_o^2 = \frac{10^9 \times 10^5}{2{\cdot}5 \times 1{\cdot}5} = \frac{10^{14}}{3{\cdot}75} : \underline{\omega_0 = 5{\cdot}15 \times 10^6}$$

$$\omega_i = \omega_o\,\frac{1}{\sqrt{1 - K}} = \underline{6{\cdot}3 \times 10^6} : \omega_{ii} = \omega_o\,\frac{1}{\sqrt{1 + K}} = \underline{4{\cdot}44 \times 10^6}$$

7.—Find the input impedance and the resonance frequencies of the circuit system shown in the accompanying diagram when $C_1 = C_2$ and $L_1 = L_2$.

ANSWER.—The admittance of the component ① is

$$\mathfrak{Y}_1 = j\frac{1}{\left(\omega L_1 - \dfrac{1}{\omega C_1}\right)} = -j\frac{\omega C_1}{\omega^2 L_1 C_1 - 1}$$

The admittance of the capacitance C_M is

$$\mathfrak{Y}_M = j\,\omega\,C_M$$

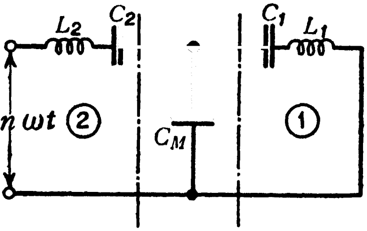


Fig. 8

The admittance of the parallel system ②: C_M is therefore

$$\mathfrak{Y}_M + \mathfrak{Y}_1 = j\omega\left[C_M - \frac{C_1}{\omega^2 L_1 C_1 - 1}\right]$$

and the impedance of these two parallel branches is

$$\mathfrak{Z}_{M+1} = \frac{1}{\mathfrak{Y}_M + \mathfrak{Y}_1} = -j\frac{\omega^2 L_1 C_1 - 1}{\omega\{\omega^2 L_1 C_1 C_M - C_M - C_1\}}$$

The input impedance is therefore

$$\mathfrak{Z}_i = \mathfrak{Z}_2 + \mathfrak{Z}_{M+1} = j\frac{(\omega^2 L_1 C_1 - 1)\,[\omega^2 L_1 C_1 C_M - (C_M + 2C_1)]}{\omega C_1\,[\omega^2 L_1 C_1 C_M - (C_M + C_1)]}$$

The input impedance becomes infinite (i.e. parallel resonance) when

$$\omega^2 L_1 C_1 C_M = C_M + C_1$$

that is

$$\boxed{\omega_0{}^2 = \frac{C_M + C_1}{L_1 C_1 C_M} = \frac{1}{L_1 C_R}}$$

where $C_R = \dfrac{C_1 C_M}{C_1 + C_M}$ and is the equivalent capacitance of C_1 and C_M in series. That is to say, ω_0 is the resonance frequency of the two branches C_M and ① in series.

The input impedance becomes zero (i.e. series resonance) for two separate conditions, viz.: (i) $\omega^2 L_1 C_1 = 1$: and (ii) $\omega^2 L_1 C_1 C_M = C_M + 2C_1$.

Case (*i*). The coupling coefficient is $K = \frac{C_1}{C_1 + C_M}$ so that $C_1 = \frac{C_M K}{1-K}$:

$$C_M = \frac{1-K}{K} C_1 : C_R = \frac{C_1 C_M}{C_1 + C_M} = K C_M$$

$$\omega^2 = \frac{1}{L_1 C_1} = \omega_0{}^2 \frac{C_R}{C_1} = \omega_0{}^2 K \frac{C_M}{C_1}$$

or

$$\underline{\omega_i = \omega_0 \sqrt{1-K}}$$

Case (*ii*)

$$\omega^2 = \frac{C_M + 2C_1}{L_1 C_1 C_M} = \omega_0{}^2 C_R \left(\frac{C_M + 2\,C_1}{C_1 C_M}\right) = \omega_0{}^2 K \left[\frac{1-K}{K} + 2\right]$$

that is,

$$\underline{\omega_{ii} = \omega_0 \sqrt{1+K}}$$

The respective resonance frequencies as defined by Cases (i) and (ii) may then be summarised :—

$$\boxed{\left.\begin{matrix}\omega_i \\ \omega_{ii}\end{matrix}\right\} = \omega_0 \sqrt{1 \pm K}}$$

Thus, if $C_1 = 2{\cdot}5 \times 10^{-9} F$: $C_M = 5 \times 10^{-9} F$: $L_1 = 10^{-5} H$.

$$C_R = \left(\frac{2{\cdot}5 \times 5}{2{\cdot}5 + 5}\right) 10^{-9} = 1{\cdot}67 \times 10^{-9} F : K = \frac{2{\cdot}5}{7{\cdot}5} = 0{\cdot}333$$

$$\underline{\omega_0} = \sqrt{\frac{10^{14}}{1{\cdot}67}} = \underline{7{\cdot}76 \times 10^6}$$

$$\omega_i = \omega_0 \sqrt{1-K} = 0{\cdot}815\ \omega_0 = \underline{6{\cdot}34 \times 10^6}$$

$$\omega_{ii} = \omega_0 \sqrt{1+K} = 1{\cdot}15\ \omega_0 = \underline{8{\cdot}9 \times 10^6}$$

8.—*Explain the action of a simple valve oscillator.*

Answer.—In Fig. 9 is shown an oscillatory circuit of known constants, this parallel system being connected across the grid and filament of the valve. Coupled by the mutual inductance of M henry is a coil which is connected at one end to the anode plate and at the other end to the positive pole of the high tension battery. Connected in parallel with the high tension battery is a condenser C_A of relatively large capacitance (a so-called "by-pass" condenser) to provide a path for the anode alternating current which does not then have to pass through the battery.

If an alternating current is started in the oscillatory circuit, say, by means of some electric impulse (actually however, no special provision is necessary for this as the current builds up spontaneously as soon as the coupling conditions are suitable), an a.c. pressure will then appear at the grid and this will give rise to a corresponding a.c. I_A in the anode circuit, this anode current being related to the grid pressure V_G by the equation

$$I_A = V_G\, G_m \quad \dots\dots \quad \text{(i)}$$

where G_m is the mutual conductance (i.e., the "slope" of the valve : see Test Paper for Chapter II, Example 13).

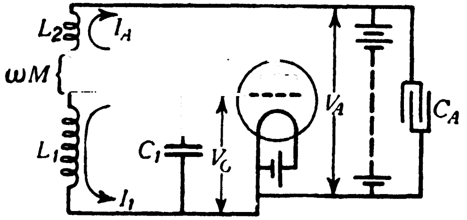


Fig. 9

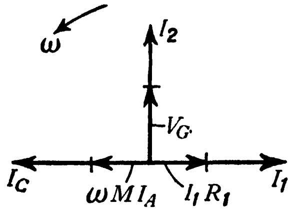


Fig. 9a

If the coils L_1 and L_2 are coupled in the correct "sense" the a.c. in L_2 will induce an e.m.f. in L_1 which will maintain the current in the oscillatory circuit if the relationships defined by the following vector equation are established, viz.,

$$\mathfrak{V}_G = j\omega L_1 I_1 + R I_1 + j\,\omega M I_A \quad \dots\dots \quad \text{(ii)}$$

and the vector diagram corresponding to this equation is shown in Fig. 2, from which it will be seen that the essential condition to satisfy (ii) is that

$$\omega M I_A = I_1 R \quad \dots\dots \quad \text{(iii)}$$

Substituting from (i) for I_A and noting that $I_1 = V_G\, \omega C$, it is seen that the condition for maintained oscillations is that

$$\omega M\, V_G\, G_m = V_G\, \omega C\, R$$

or

$$R = \frac{M}{C} G_m \quad \dots\dots \quad \text{(iv)}$$

The power loss in the oscillatory circuit is $I_1^2 R$ watts and the power injected into this circuit from the anode high tension battery is

$$I_1^2 R = (\omega M I_A) I_1 = \omega\, M\, V_G\, G_m\, I_1$$

so that $I_1^2 R = \omega M\, V_G\, G_m\, I_1$ and hence $I_1 R = \omega M\, V_G\, G_m$

that is, $V_G\, \omega C\, R = M\omega\, V_G\, G_m$

or $\boxed{R = \frac{M}{C} G_m}$ as in equation (iv).

9.—A two-pole short-circuit occurs at a power station which is operating in parallel with a number of other power stations on 10 kV. bus bars. The short-circuit current reaches a peak value of $\sqrt{2} \times 66{,}100 = 93{,}500$ amperes. If the length of conductor between two consecutive supports is $l = 160$ cm. and the mean distance between the conductors is $D = 40$ cm., find the mechanical force which will be developed between them due to the short circuit.

ANSWER.—The mechanical force P kg. of a current I r.m.s. amperes in a circuit loop of length l cm. and mean distance apart D cm. is (see Fig. 10)

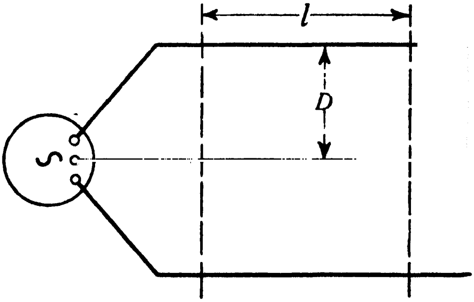


Fig. 10

$$P = 2{\cdot}04\ I_m{}^2\ \frac{l}{D}\ 10^{-8}\ \text{kg.}$$

where I_m is the peak value of the current, that is, $I_m = I\sqrt{2}$.

Inserting the given numerical data of the problem into this formula gives

$$P = 2{\cdot}04(\sqrt{2} \times 66{,}100)^2 \times \frac{160}{40} \times 10^{-8} = 720\ \text{kg.},$$

and the supporting structure for the conductor must be able to withstand this force.

10.—The natural frequency of oscillation of a stretched conductor supported at each end is

$$n = 112\ \sqrt{\frac{EI}{gl^4}}\ \text{oscillations per second,}$$

where g kg. per centimetre length is the weight of the conductor,

E is Young's modulus and for copper is 1.15×10^6 kg. per square centimetre,

J is the moment of inertia of the cross-section of the conductor in cm.^{4} units,

l cm. is the free length of the stretched conductor.

If the copper bus bars of Example 9 have a cross-section of 10×1 cm^2. and the distance between two supports is 160 cm., find (i) the natural frequency of oscillation, and (ii) the stress in the conductors due to the force of $P = 720$ kg. as calculated for the short-circuit conditions of Example 9.

ANSWER.—Assuming the two conductors are arranged side by side (Fig. 11) then the value of the moment of inertia J to be inserted in the formula for the natural frequency will be that which is taken about the middle vertical axis, as shown in Fig. 12, viz., $J = \frac{1}{12} b^3h$ where $b = 1{\cdot}0$ cm.: $h = 10$ cm. so that $J = 0{\cdot}83$ and $g = 0.09$. Hence,

$$n = 112\sqrt{\frac{1{\cdot}15 \times 10^6 \times 0{\cdot}83}{0{\cdot}09 \times 160^4}} = \frac{112}{160^2}\sqrt{\frac{1{\cdot}15 \times 0{\cdot}83}{0{\cdot}09}}\,10^3$$

$$= \frac{112 \times 10^3}{2{\cdot}56 \times 10^4}\sqrt{\frac{0{\cdot}955}{0{\cdot}09}} = 4{\cdot}37 \times 3{\cdot}26$$

that is

$$\underline{n = 14{\cdot}2} \text{ oscillations per second.}$$

If the distance between two consecutive supports were to be reduced to 100 cm., the natural frequency of oscillation would be increased in the ratio $\left(\frac{160}{100}\right)^2$ so that the frequency would then be

$$n = 2{\cdot}56 \times 14{\cdot}2 = 36{\cdot}6 \text{ oscillations per second,}$$

and this is sufficiently near to the supply frequency of 50 to enable the conductors to respond to the pulsation of the force due to the short circuit current, viz., pulsation of 100 frequency.

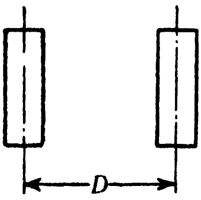


Fig. 11

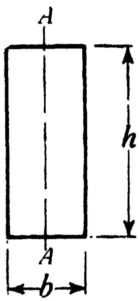


Fig. 12

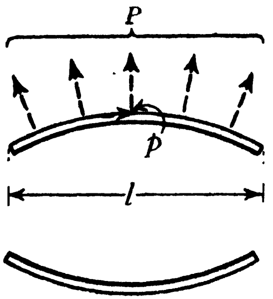


Fig. 13

The stress is given by the following equation (see Figs. 12 and 13).

$$p = \frac{M}{J}\,\frac{b}{2} = \frac{P.l \times \frac{b}{2}}{8 \times \frac{1}{12}\,b^3h}$$

where M is the bending moment

or $$p = \frac{3}{4}\,\frac{P.l}{b^2h} \text{ kg, per sq. cm.}$$

where P is in kg., and $b : h$: and l in cms.
If $b = 1$ cm. : h = 10 cm. : $l = 160$ cm.

then $p = \frac{3}{4}\,\frac{720 \times 160}{1 \times 10} = \underline{8{,}500}$ kg. per cm.^{2}

Copper, however, will break under a stress of 3,500 kg./cm^2 so that the copper sections specified cannot support the stresses due to the sudden short-circuit current when the rectangular conductors are placed side by side as shown in Fig. 11. If, however, the conductors are placed vertically in line as shown in Fig. 14 the moment of inertia will be taken about the axis BB of Fig. 15, so that $J = \frac{1}{12}\,h^3b$ and consequently

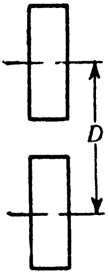


Fig. 14

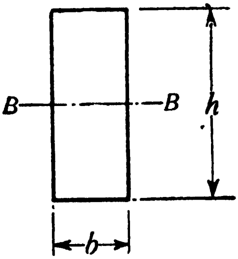


Fig. 15

$$p = \frac{1}{8}\cdot\frac{Pl\,\frac{h}{2}}{\frac{1}{12}\,h^3b} = \frac{3}{4}\cdot\frac{Pl}{h^2b} = \frac{3}{4}\cdot\frac{720 \times 160}{1 \times 100} = \underline{850 \text{ kg}}. \text{ per cm.}^2$$

11.—A transformer connects a 30-kV. supply cable with a 6-kV. network as shown in the accompanying diagram a circuit-breaker being installed on each side of the transformer. The three-phase 30/6-kV. transformer is rated at 15 MVA., the leakage reactance is 7.7 per cent., no-load current (i.e. magnetising current) is 7.1 per cent. of the rated full-load current. The low-tension switch opens by means of an arc and it is required to find the oscillation frequencies to which it will give rise.

Answer.—The circuit system is shown in Fig. 16. The leakage reactance is given by

$$\omega L_S\, I_N = 0{\cdot}077\,\frac{V_l}{\sqrt{3}}$$

Where I_N is the normal full load current per phase and V_l volts the line pressure, that is

$$I_N = \frac{15 \times 10^6}{\sqrt{3}\, V_l}$$

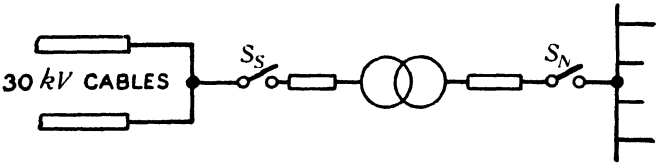


Fig. 16

so that
$$L_S = \frac{0{\cdot}077}{\omega}\cdot\frac{V_l}{\sqrt{3}}\cdot\frac{\sqrt{3}\, V_l}{15 \times 10^6} = \frac{0{\cdot}077\, V_l^2}{\omega\, 15 \times 10^6}$$

$$V_l = 30 \times 10^3 : \omega = 314 :: \therefore L_S = \underline{14{\cdot}7 \text{ millihenrys.}}$$

Also

$$I_O = 0{\cdot}071\, I_N = 0{\cdot}071\,\frac{W_N}{\sqrt{3}\, V_l} = 0{\cdot}071\,\frac{15 \times 10^6}{(\sqrt{3} \times 30 \times 10^3)} = \underline{20{\cdot}5 \text{ amperes.}}$$

$$\omega\, L_O\, I_O = \frac{V_l}{\sqrt{3}} : L_O = \frac{30 \times 10^3}{\sqrt{3}\,\omega\, 20{\cdot}5} = \underline{2{\cdot}7 \text{ henrys.}}$$

$$L = L_N + L_S = (1{\cdot}6 + 14{\cdot}7) \text{ millihenrys.}$$
$$= \underline{16{\cdot}3 \text{ millihenrys.}}$$

When the switch on the low tension side of the transformer is opened through an arc, an impulse disturbance is given to the system and gives rise to a transient oscillation in the circuit C, L_S, L_N, C_N (Fig. 17). Since the capacitance C_N is relatively so large that its reactance can be taken to be negligible, the natural frequency of oscillation will be

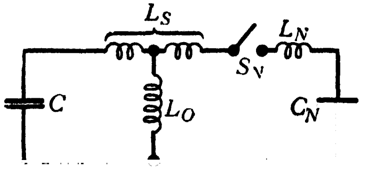


Fig. 17

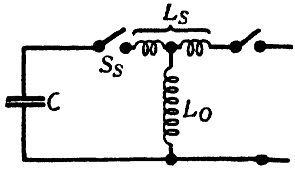


Fig. 18

$$f_h = \frac{1}{2\pi}\,\frac{1}{\sqrt{C\,(L_N + L_S)}} = \frac{1}{2\pi}\,\frac{1}{\sqrt{0{\cdot}06 \times 10^{-6} \times 16{\cdot}3 \times 10^{-3}}}\ \text{hertz.}$$

$$= \frac{1}{2\pi}\sqrt{\frac{10^9}{0{\cdot}98}} = \frac{10^5}{2\pi}\sqrt{\frac{1}{9{\cdot}8}} = \underline{5{,}060\ \text{hertz.}}$$

that is to say, this will be a relatively high frequency oscillation.

The circuit-breaker arc will become quenched as the load current passes through its zero value and if the high tension circuit-breaker then begins to open, the interruption of the magnetising current will start a transient current in the circuit of C and L_O in Fig. 18, and since the inductance L_O is relatively very large, the frequency of this transient will be relatively low, viz.,

$$f_e = \frac{1}{2\pi}\cdot\frac{1}{\sqrt{L_O\,C}} = \frac{1}{2\pi}\cdot\frac{1}{\sqrt{2{\cdot}7 \times 0{\cdot}06 \times 10^{-6}}}$$

$$= \frac{10^3}{2\pi\sqrt{0{\cdot}162}} = \frac{10^3}{2{\cdot}53} = \underline{395\ \text{hertz.}}$$

If the circuit-breaker interrupts the magnetising current at the moment when it is passing through its peak value I_{om} the electromagnetic energy stored in the inductance will be:

$$\tfrac{1}{2}\,I^2_{OM}\,L_O = \tfrac{1}{2}\,(\sqrt{2}\,I_O)^2\,L_O = \tfrac{1}{2} \times 2 \times 20{\cdot}5^2 \times 2{\cdot}7\ \text{joules}$$
$$= 420 \times 27 = \underline{1{,}130\ \text{joules.}}$$

This energy will oscillate from the inductance L_O to the capacitance C and when it has all passed into the capacitance the p.d. across the capacitance will be E_O where

$$\tfrac{1}{2}\,C\,E_O^2 = \tfrac{1}{2}\,I^2_{OM}\,L_O = 1{,}130\ \text{joules.}$$

so that $$E_O = \sqrt{\frac{1{,}130 \times 2}{C}} = \sqrt{\frac{1{,}130 \times 2 \times 10^6}{0.06}}$$

that is $$\underline{E_O = 196{,}000\ \text{volts.}}$$

12.—Three simple series circuits are respectively defined by the following data, viz.:

(i) $C = 200\ \mu F.$: $L = 0.1$ henry: $R = 5$ ohms
(ii) $C = 200\ \mu F$: $L = 0.1$ henry: $R = 0$ ohms
(iii) $C = 40\ \mu F$: $L = 0.5$ henry: $R = 0$ ohms

Draw the graphs showing the relationship between current and supply frequency for the following arrangements of these circuits:

I. The simple series circuit 1 is alone connected across the supply terminals.

II. The circuits 1 and 2 are connected in series across the supply terminals.

III. The circuits 1 and 3 are connected in series across the supply terminals.

In each case the peak value of the supply pressure is 1,000 volts.

ANSWER.—

For the system I, Fig. 19 : $C_1 = 200 \mu F$; $L_1 = 0.1\ H$; $R_1 = 5$ ohms so that the natural frequency is :

$$\omega_0 = \frac{1}{\sqrt{L_1 \times C_1}} = \frac{10^3}{\sqrt{20}} = 224$$

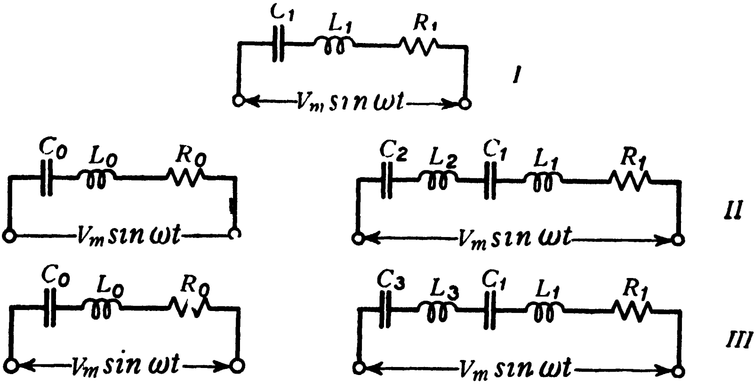


Fig 19

For the system II : The equivalent simple series circuit as shown in Fig. 19 on the left-hand side of *II*, will have the following constants, viz. :

$$C_0 = \frac{C_1 \times C_2}{C_1 + C_2} = \frac{200 \times 200}{400} = 100 \mu F,$$

$$L_0 = L_1 + L_2 = 0.2\ H,$$

$$R_0 = R_1 = 5 \text{ ohms}$$

so that the natural frequency is :

$$\omega_0 = \frac{1}{\sqrt{L_0 \times C_0}} = \frac{10^3}{\sqrt{20}} = 224.$$

For the system III : The equivalent simple series circuit shown on the left-hand side of *III* in Fig. 19, will have the following constants, viz. :

$$C_0 = \frac{C_1 \times C_3}{C_1 + C_3} = \frac{200 \times 40}{240} = \frac{1000}{30} \ \mu F$$

$$L_0 = L_1 + L_3 = 0.6H$$

$$R_0 = R_1 = 5 \text{ ohms}$$

so that the natural frequency is :

$$\omega_0 = \frac{1}{\sqrt{L_0 \times C_0}} = \frac{10^3}{\sqrt{20}} = 224.$$

It will be seen, therefore, that each of the circuits (i), (ii) and (iii) have the same natural frequency, viz., $\omega_0 = 224$.

The graph I of Fig. 20 shows the relationship of current and supply frequency for the circuit (i), for an applied pressure of 1,000 volts peak value.

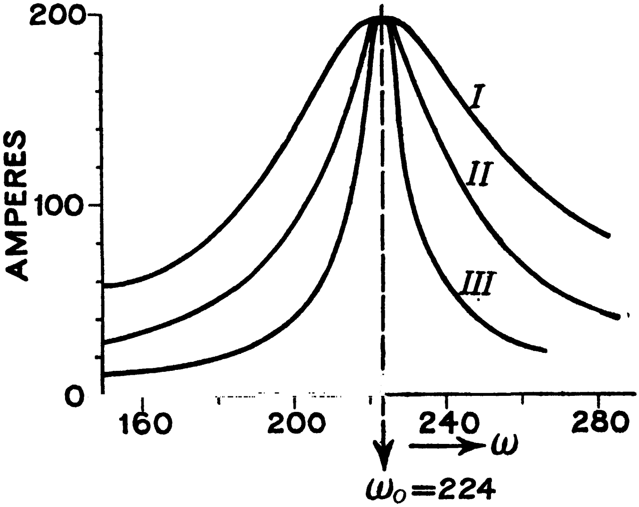


Fig. 20

When the circuits (i) and (ii) are connected in series, the natural frequency of this system will still be $\omega_0 = 224$, and in the graph II of Fig. 20 is shown the relationship between the current and supply frequency for an applied pressure of 1,000 volts peak value, and it will be seen that the resonance is much sharper than for the graph I.

When the circuits (i) and (iii) are placed in series, it will be seen that the natural frequency is again $\omega_0 = 224$, and graph III of Fig. 20 shows the current frequency relationship for this case, the resonance being much sharper than in either II and I.

These results illustrate the general formula for the factor damping, viz. :

$$\beta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

which shows that *small capacitance and large inductance are favourable for small damping*. (See also Example 5, Chapter XVI.)

For the system *III* for example, it will be noted that at absolute resonance

$$V_m = RI_m = 5 \times 200 = \underline{\underline{1000 \text{ volts,}}}$$

and the pressure across the inductance L_o will be

$$V_{L_o} = \omega_o \, L_o \, I_m = 224 \times 0{\cdot}6 \times 200$$
$$= \underline{\underline{26{,}880 \text{ volts,}}}$$

so that

$$Q = \frac{V_{L_o}}{V_R} = \frac{\omega_o L_o}{R} = \frac{224 \times 0{\cdot}6}{5} = \underline{\underline{26.88}}$$

and in radio technique this is termed the " amplification factor."

13.—Show how alternating current of constant frequency may be generated by means of a mechanical oscillating system.

ANSWER.—For generating alternating current at frequencies up to 10^4 hz., the vibrations of a tuning-fork of corresponding frequency may be used. When a tuning-fork is vibrating, the frequency of its oscillations are maintained constant to a remarkably high degree and, by means of an electromagnet arrangement, the tuning-fork may be kept oscillating at its natural frequency for an indefinitely long time.

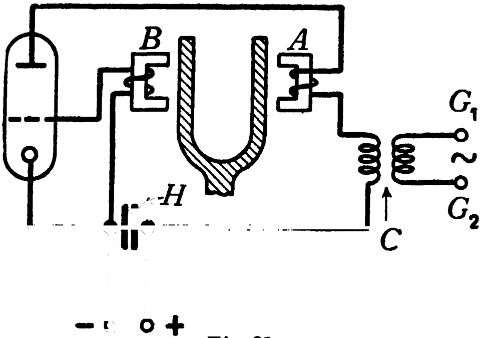


Fig. 21

In Fig: 21 is shown a circuit arrangement by means of which the energy of the mechanical oscillations can be converted into the electrical energy of alternating current of the corresponding frequency. This circuit comprises the back-coupling of the valve anode to the grid, and the alternating current generated in this way passes through the exciting coil of the electromagnet *A* and so maintains the tuning-fork in persistent vibration, whilst the electromagnet *B* applies the required back-coupling pressure to the grid of the valve : a blocking condenser is shown at *H*. Looked at from another point of view (see also Test Paper, Chapter II, Example 13), the e.m.f. developed by the electromagnet *B* and applied to the grid of the valve is amplified by the valve and the alternating pressure so obtained sends an alternating current through the primary winding of the transformer *C*. In this way the alternating pressure of the required constant frequency becomes available at the terminals G_1 G_2.

The constancy of the vibration frequency is remarkably high : commercial tuning-forks of steel have a temperature coefficient of -1×10^{-4} and if special steel is used this can be reduced to the value 10^{-6}. Further, there is no special difficulty in building the tuning-fork into a thermostat structure in which the temperature can be maintained constant to 0·01°C. In this way the frequency of the tuning-fork may be maintained constant to about $\frac{1}{10^8}$ of its specified value. This type of frequency generator was first proposed by W. H. Eccles in 1919 and since then has been widely used for providing a standard of frequency for radio-networks.

14.—*Give a short account of the vibrating-reed rectifier.*

Answer.—For many years great efforts have been made to provide a satisfactory appliance for measuring small alternating current quantities, by

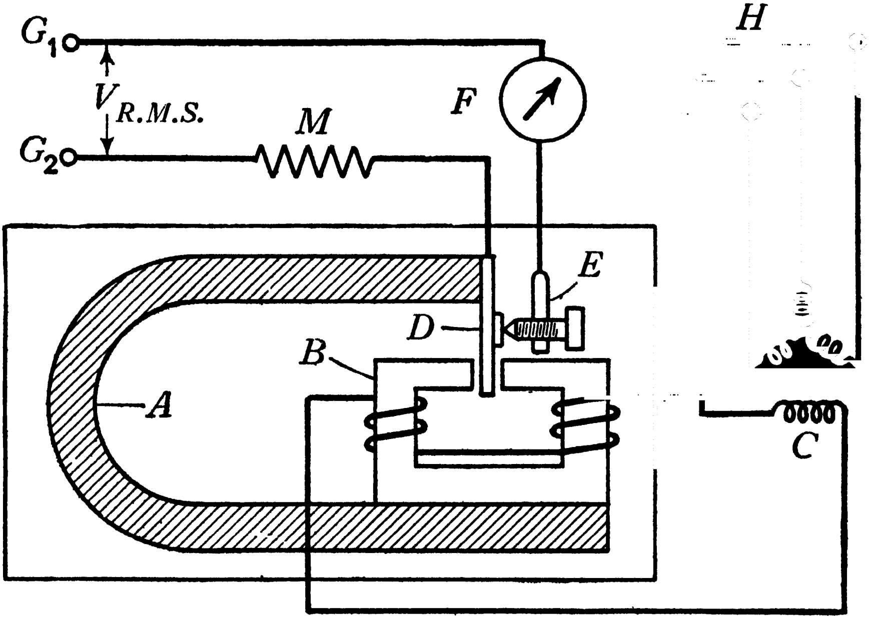


Fig. 22

means of indicating instruments which have all the favourable characteristics of the moving-coil d.c. type of instrument, viz., small power requirements, high sensitivity, and a linear scale. This problem has now been solved in a very satisfactory manner by means of a rectifier, the basic principle of which is a mechanically oscillating contact. This type of rectifier possesses ideally suitable features, viz., infinitely high resistance for (say) the negative half-wave of the alternating current, and approximately zero contact resistance for the positive half-wave.

The general principle of this rectifier will be clear by referring to Fig. 22. The permanent magnet *A* supports a laminated electromagnet system *B*, the excitation coils of which are supplied with alternating current by means of the induction regulator *C*. The tuned vibratory reed *D* carries the flux of the permanent magnet and superposed on this flux is the alternating flux due to the excitation of the electromagnet *B*, so that there will be an oscillatory magnetic force tending to vibrate the reed *D*. If the natural frequency of the reed is tuned to the frequency of this oscillating force, the reed will be maintained in persistent vibration and will make and break contact with the fixed stop attached to the support *E*. If the a.c. supply connected to the terminals G_1 and G_2 is of the correct phase (as may be obtained by adjustment of the induction regulator *C*), and is also in synchronism with the vibrating reed, a rectified current will pass through the d.c. moving-coil instrument *F* shown in Fig. 22. If the a.c. pressure at the terminals G_1, G_2 is sinusoidal, the instrument reading will be a measure of the magnitude of the a.c. pressure. Thus, if $I_{\text{d.c.}}$ is the reading of the d.c. instrument, this quantity will be related to the peak value $I_{\text{max.}}$ of the rectified current (Fig. 22) as follows :

$$I_{\text{d.c.}} = \frac{1}{2}\frac{2}{\pi} I_{\text{max.}} = \frac{1}{\pi} I_{\text{max.}}$$

and the corresponding r.m.s. value $V_{\text{r.m.s.}}$ of the pressure across the terminals $G_1\ G_2$ will then be given by the relationships :

$$V_{\text{r.m.s.}} = \frac{V_{\text{max.}}}{\sqrt{2}} = \frac{1}{\sqrt{2}} I_{\text{max.}} \times R = \frac{\pi}{\sqrt{2}} I_{\text{d.c.}} \times R$$

Where *R* is the resistance of the circuit of the instrument *F* in Fig. 22.

One practical example of a vibrating reed type of rectifier may be mentioned, one having a rating of 5 mA. at 5 volts with a permissible short-time overload of 100 mA. at 10 volts. The excitation of the equipment requires 5 watts and the accuracy is within ± 0.5 per cent.

15.—Show how a quartz crystal may be used to stabilise the frequency of a valve generator.

ANSWER.—The electric behaviour of a piezo-electric crystal plate (Fig. 23) can be represented by an equivalent circuit comprising resistance, inductance, and capacitance components. The alternating current which flows through a crystal plate when oscillating in service may be regarded as consisting of two components, viz., a current I_1 due to the electron polarisation of the dielectric (which effect is practically without inertia) and is defined by the dimensions of the plate and the dielectric constant. As regards this current component, the crystal may be replaced by a capacitance

$$C_1 = \frac{\varepsilon . A}{4\pi . d}.$$

where ε is the dielectric constant, A is the surface area and d the thickness of the plate.

The second current component I_2 is due to the mechanical displacement of the electric charge carriers inside the plate and this effect is not without inertia but is defined by the mechanical and elastic properties of the material of the plate. With respect to this current component, the crystal corresponds to a very slightly damped series oscillatory circuit consisting of a small capacitance C_{cr}, a large inductance L_{cr}, and a small damping resistance R_{cr}. If the electrodes S_1 and S_2, Fig. 23, do not press directly on the surfaces of the quartz plate Q then both component currents I_1 and I_2 must flow across the corresponding air-gap, that is, through a capacitance C_2. In parallel with the whole system is the capacitance C_3 of the supporting structure and leads of the quartz plate. It can be shown that the whole system may be replaced by the simple circuit of Fig. 24 and, in what follows, this circuit will be used as the basic circuit diagram for the oscillating quartz plate.

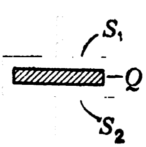


Fig. 23

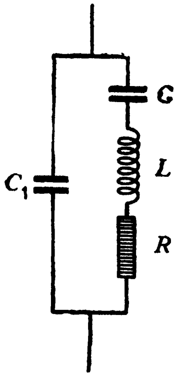


Fig. 24

The type of circuit shown in Fig. 24 will have two distinct resonance conditions, viz., a " series " (or pressure) resonance, and a " parallel " (or current) resonance. The frequency for the series resonance is only dependent upon the quantities R, L, and C of the circuit of Fig. 24 whilst the frequency for parallel resonance will depend upon the capacitance C_1 also. Thus:

For series resonance: $$\omega_s = \frac{1}{\sqrt{L.C}}$$

For parallel resonance: $$\omega_p = \frac{1}{\sqrt{L \times \left(\frac{C.C_1}{C + C_1}\right)}}$$

Parallel resonance will appear at a higher value of the frequency than series resonance. Since, however, C_1 will be much greater than C, the two resonance frequencies will be, in general, not far removed from each other.

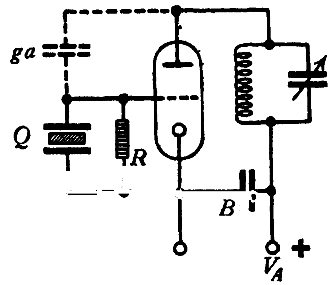


Fig. 25

For the stabilisation of the frequency of a valve generator by means of a piezo-electric crystal, many circuit systems have been devised, the best known of which is the Pierce-circuit shown in Fig. 25. In this circuit the back coupling is obtained by the grid-anode capacitance C_{ga} as is shown by the broken-line part of the diagram. Oscillations will only be generated if the grid-cathode path as well as the external impedance is inductive.

In Fig. 26 the resistance of the crystal circuit system is shown as a function of the frequency, extending from the value ω_s for series resonance to the value ω_p for parallel resonance, and beyond.

The resonance frequency for the transverse oscillations of a plate such as that shown in Fig. 23 depends upon the thickness of the plate and on its elastic characteristics. That is,

$$N = \nu d$$

where N is the oscillation coefficient. For the plate of the orientation shown in Fig. 23 the oscillation coefficient is $1{\cdot}69 \times 10^6$ hz. per mm. and the wavelength of the oscillations is 116 metre/mm.

The significance of the quartz piezo-electric effect is due to the extremely sharp resonance characteristic as shown in Fig. 26, and this is accounted for by the remarkably small internal damping of the oscillating plate. When placed in an oscillating electric field of appropriate frequency the mechanical oscillations of the crystal at resonance may, in consequence of the extremely small damping, attain such a magnitude as to shatter the crystal.

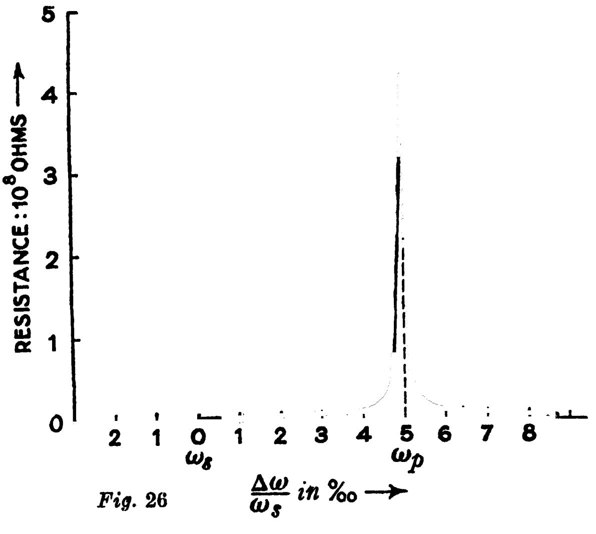


Fig. 26

Since the primary characteristic of the crystal oscillator, and particularly of the quartz crystal, depends upon the extreme sharpness of the peak of the resonance curve (Fig. 26), the quartz plate as used for transverse oscillation is mounted on supports such that no supplementary damping is produced. Circular quartz plates are held by supports which grip the central layer of the plate. For this purpose, either a circular groove can be cut of depth sufficient to reach the central layer and holding screws then arranged to grip this layer, or the periphery of the plate may be chamfered down on each side so that the central layer protrudes as a knife-like edge to which the support attachments may be fixed.

A source of additional damping is the air friction, especially in such cases where an air cushion between the plate and the electrodes is brought into oscillation. For this reason, the quartz oscillator is sometimes arranged in a vacuum and the outer appearance of such a construction is similar to that of a small valve generator. A double wall is provided similar to that of a Dewar vacuum flask and a special heating device is fitted. In order to obtain the exact setting of the frequency of the plate, either adjustment is made of the distance between the electrodes, or, when the vacuum fitting is provided, the frequency can be adjusted by control of the temperature. For very great constancy of frequency the temperature can be maintained constant by means of thermostat control. Since the temperature coefficient of the quartz is only 1×10^{-6}, it is generally sufficient, even for exact requirements, to maintain the temperature constant within $^1/_{10}{}^\circ$ C. and this can be done without much difficulty.

Of the multitude of piezo-electric crystals available the only ones which have so far found practical application are quartz, tourmaline, and Seignette salts (i.e. Rochelle salts). Of these, quartz is distinguished by its extremely low temperature coefficient, which may be even less than 1×10^{-6}. Tourmaline has found application in service with short-wave valve generators for wave-lengths of about $\lambda = 8$ metres.

Chapter XI

ALTERNATING CURRENT POWER

1.—Explain the principle of action of some form of relay by means of which the power factor of the current which is being transmitted from one station to another may be automatically controlled.

ANSWER.—The control of the power factor at which the three-phase current is being sent from one station to another, can be made automatic by means of a relay which operates on the principle of the wattmeter type of indicating instrument, in which a force is developed between a “ pressure coil ” and a “ current coil.”

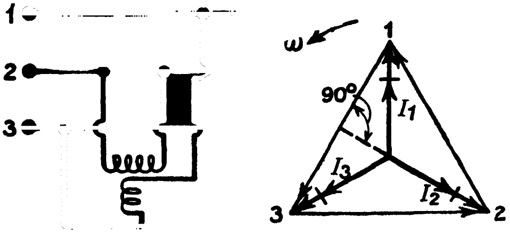


Fig. 1

(i) Suppose, for example, it is required to maintain the transmitted current at unity power factor. In Fig. 1 is shown a diagram for which the current coil is in line 2 of the supply and the pressure coil is across lines 1 and 3.

It will be seen from the vector diagram of Fig. 1 that for unity power factor (i.e., $\phi = 0$), the current in the pressure coil will be 90° out of phase with that in the current coil, so that no torque will be developed on the moving part of the relay under these conditions. If, however, current begins to flow with a phase displacement, $\phi > 0$, a torque will be developed, and the consequent movement of the pressure coil will close a contact and so actuate the regulating mechanism necessary to restore the power factor to unity.

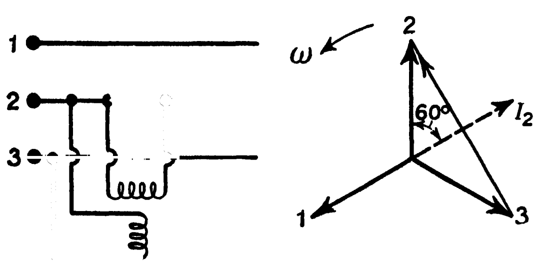


Fig. 2

(ii) Suppose now that it is required to maintain the power factor constant at the value $\phi=0.5$ lagging. The connections shown in Fig. 2 will correspond to the condition that no torque will be developed on the relay movement when the power factor is cos $\phi = 0.5$ lagging. The connections shown in Fig. 3 correspond to no torque when the power factor is 0.5 leading.

In a similar way, a suitable circuit connection can be arranged to give no torque for any given value of the power factor and thus, any departure from the prescribed power factor will actuate the regulating mechanism until the normal condition has become restored.

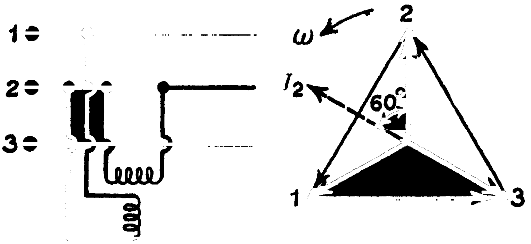


Fig. 3

2.—A symmetrical three-phase supply of which the line pressure is 380 volts feeds a mesh connected load as follows:

Between lines 1 and 2: 19 kVA. at cos φ = 0.5 lagging.
" " 2 " 3: 30 kVA. at cos φ = 0.8 lagging.
" " 3 " 1: 10 kVA. at cos φ = 0.9 leading.

Draw the vector diagram for the mesh loads and derive the vector diagram for the line currents.

ANSWER.—The mesh-connected load is shown in Fig. 4. The magnitudes of the loads currents are—

Between lines 1 and 2: $I_{12} = \dfrac{19000}{380} = 50$ amperes.

" " 2 and 3: $I_{23} = \dfrac{30000}{380} = 79$ amperes.

" " 3 and 1: $I_{31} = \dfrac{10000}{380} = 26{\cdot}4$ amperes.

The mesh impedances are obtained as follows—

$$I_{12}R_{12} = V_{12} \cos\phi : R_{12} = \frac{380 \times 0{\cdot}5}{50} = 3{\cdot}8\Omega$$

$$I_{12}X_{12} = V_{12} \sin\phi : X_{12} = \frac{380 \times 0{\cdot}866}{50} = 6{\cdot}6\Omega$$

so that

$$\mathfrak{Z}_{12} = 3{\cdot}8 + j6{\cdot}6 = 7{\cdot}6e^{j58^\circ}\Omega$$

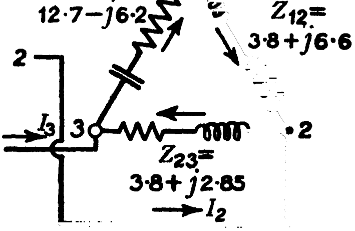


Fig. 4

Similarly

$$\mathfrak{Z}_{23} = 3{\cdot}8 + j2{\cdot}85 = 4{\cdot}72e^{j37^\circ}\Omega$$
$$\mathfrak{Z}_{31} = 12{\cdot}7 - j6{\cdot}2 = 14{\cdot}1e^{-j26^\circ}\Omega$$

In Fig. 5 is shown the vector diagram for the mesh load. In order to obtain the line currents the following relationships hold:

$$\mathfrak{I}_1 = \mathfrak{I}_{12} - \mathfrak{I}_{31} : \mathfrak{I}_2 = \mathfrak{I}_{23} - \mathfrak{I}_{12} : \mathfrak{I}_3 = \mathfrak{I}_{31} - \mathfrak{I}_{23}$$

and these line currents are shown by the broken line vectors of Fig. 5.

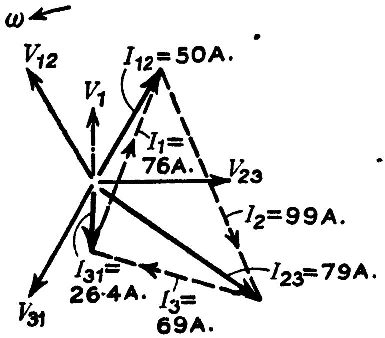


Fig. 5

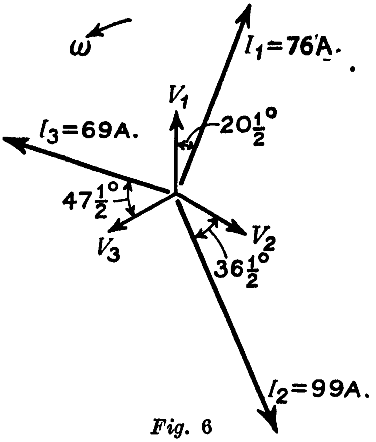


Fig. 6

In Fig. 6 the vectors of line currents and the phase vectors of pressure are shown. The vectors of line current are as follows (referred to the phase pressure vector $\mathfrak{V}_1$ as the datum), viz.:

$$\mathfrak{I}_1 = \underline{76e^{-j20\frac{1}{2}^\circ}} : \mathfrak{I}_2 = \underline{99e^{-j156\frac{1}{2}^\circ}} : \mathfrak{I}_3 = \underline{69e^{j72\frac{1}{2}^\circ}}$$

3.—Convert the mesh load of Example 2 into the equivalent star load. Find the pressure of the load star-point as referred to the star-point of the supply system and derive the vector diagram for the line currents.

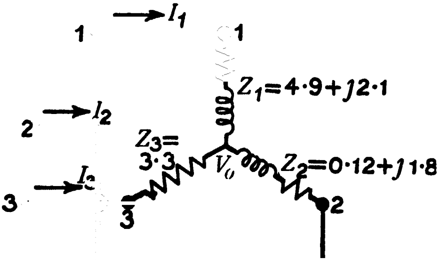


Fig. 7

Answer.—The equivalent star load is given by the following expressions, (see Fig. 7):

$$\mathfrak{Z}_1 = \frac{\mathfrak{Z}_{12}\mathfrak{Z}_{31}}{\mathfrak{Z}_{12} + \mathfrak{Z}_{23} + \mathfrak{Z}_{31}} = \frac{7{\cdot}6 \times 14{\cdot}1e^{j32}}{20{\cdot}2e^{j9^\circ}} = 5{\cdot}3e^{j23^\circ} = 4{\cdot}9 + j2{\cdot}1$$

$$\mathfrak{Z}_2 = \frac{\mathfrak{Z}_{12}\mathfrak{Z}_{23}}{\mathfrak{Z}_{12} + \mathfrak{Z}_{23} + \mathfrak{Z}_{31}} = \frac{7{\cdot}6 \times 4{\cdot}72e^{j95^\circ}}{20{\cdot}2e^{j9^\circ}} = 1{\cdot}8e^{j86^\circ} = 0{\cdot}12 + j1{\cdot}8$$

$$\mathfrak{Z}_3 = \frac{\mathfrak{Z}_{31}\mathfrak{Z}_{23}}{\mathfrak{Z}_{12} + \mathfrak{Z}_{23} + \mathfrak{Z}_{31}} = \frac{14{\cdot}1 \times 4{\cdot}72e^{j11^\circ}}{20{\cdot}2e^{j9^\circ}} = 3{\cdot}3e^{j2^\circ} \simeq 3{\cdot}3$$

To obtain the pressure $\mathfrak{V}_o$ of this load star-point (see Fig. 7):

$$\mathfrak{I}_1 = \frac{\mathfrak{V}_1 - \mathfrak{V}_o}{\mathfrak{Z}_1} : \mathfrak{I}_2 = \frac{\mathfrak{V}_2 - \mathfrak{V}_o}{\mathfrak{Z}_2} : \mathfrak{I}_3 = \frac{\mathfrak{V}_3 - \mathfrak{V}_o}{\mathfrak{Z}_3}$$

and since

$$\mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3 = 0 \text{ it follows that}$$

$$\frac{\mathfrak{V}_1}{\mathfrak{Z}_1} + \frac{\mathfrak{V}_2}{\mathfrak{Z}_2} + \frac{\mathfrak{V}_3}{\mathfrak{Z}_3} = \mathfrak{V}_o \left\{ \frac{1}{\mathfrak{Z}_1} + \frac{1}{\mathfrak{Z}_2} + \frac{1}{\mathfrak{Z}_3} \right\}$$

Putting

$$e^{j120^\circ} = e^{j2\pi/3} = a : e^{j240^\circ} = a^2 :$$

then

$$\mathfrak{V}_1 \left\{ \frac{1}{\mathfrak{Z}_1} + \frac{a^2}{\mathfrak{Z}_2} + \frac{a}{Z_3} \right\} = \mathfrak{V}_o \left\{ \frac{1}{\mathfrak{Z}_1} + \frac{1}{\mathfrak{Z}_2} + \frac{1}{\mathfrak{Z}_3} \right\}$$

that is

$$\mathfrak{V}_1 \left\{ \frac{e^{-j23^\circ}}{5{\cdot}3} + \frac{e^{j154^\circ}}{1{\cdot}8} + \frac{e^{j120^\circ}}{3{\cdot}3} = \mathfrak{V}_o \left\{ 0{\cdot}175 - j0{\cdot}07 + 0{\cdot}04 - j0{\cdot}55 + 0{\cdot}3 \right\} \right.$$

so that

$$\mathfrak{V}_1 \{ -0{\cdot}47 + j0{\cdot}43 \} = \mathfrak{V}_o \{ 0{\cdot}515 - j0{\cdot}62 \}$$

and hence, after reducing to the simplest form,

$$\mathfrak{V}_o = \mathfrak{V}_1 \{ -0{\cdot}79 - j0{\cdot}11 \} = 0{\cdot}8\, \mathfrak{V}_1 e^{j187^\circ} = \underline{176e^{j187^\circ}} \text{ volts}$$

therefore,

$$\mathfrak{I}_1 = \frac{\mathfrak{V}_1 - \mathfrak{V}_o}{\mathfrak{Z}_1} = 0{\cdot}34\mathfrak{V}_1 e^{-j19^\circ} = \underline{75e^{-j19^\circ} \text{ amperes}}$$

Similarly

$$\mathfrak{I}_2 = \frac{\mathfrak{V}_2 - \mathfrak{V}_o}{\mathfrak{Z}_2} = \underline{101e^{-j155^\circ} \text{ amperes}}$$

$$\mathfrak{I}_3 = \frac{\mathfrak{V}_3 - \mathfrak{V}_o}{\mathfrak{Z}_3} = \underline{68e^{j73\frac{1}{2}^\circ} \text{ amperes}}$$

These results are seen to be in good agreement with those obtained in Example 2, of this Test Paper.

4.—An unsymmetrical three-phase pressure system is connected to a star arrangement of reactances, all three reactances being of equal magnitude. Show that the star-point pressure of the reactance will lie at the centre of gravity of the supply line pressure triangle.

ANSWER.—The star arrangement of reactances is shown in Fig. 8, the magnitude of each reactance being X ohms. Then,

$$\text{the pressure across } aO_X = \mathfrak{V}_T = jX\mathfrak{I}_T$$
$$\text{,, ,, } bO_X = \mathfrak{V}_S = jX\mathfrak{I}_S$$
$$\text{,, ,, } cO_X = \mathfrak{V}_R = jX\mathfrak{I}_R$$

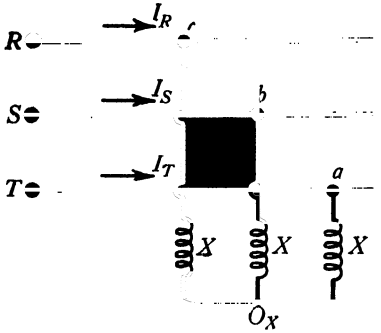


Fig. 8

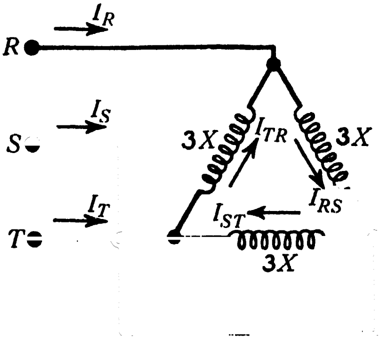


Fig. 9

If the star system of reactances is transformed into the equivalent mesh system as shown in Fig. 9, then

$$\mathfrak{I}_R = \mathfrak{I}_{RS} - \mathfrak{I}_{TR} = \frac{1}{3jX}(\mathfrak{E}_{RS} - \mathfrak{E}_{TR})$$

$$\mathfrak{I}_S = \mathfrak{I}_{ST} - \mathfrak{I}_{RS} = \frac{1}{3jX}(\mathfrak{E}_{ST} - \mathfrak{E}_{RS})$$

$$\mathfrak{I}_T = \mathfrak{I}_{TR} - \mathfrak{I}_{ST} = \frac{1}{3jX}(\mathfrak{E}_{TR} - \mathfrak{E}_{ST})$$

so that from Fig. 10, which is the line pressure triangle,

$$\mathfrak{I}_R = \frac{1}{3jX}\left[2MR\right] = \frac{2}{3jX}MR = \frac{1}{jX}O_XR$$

$$\mathfrak{I}_S = \frac{1}{3jX}\left[2NS\right] = \frac{2}{3jX}NS = \frac{1}{jX}O_XS$$

$$\mathfrak{I}_T = \frac{1}{3jX}\left[2LT\right] = \frac{2}{3jX}LT = \frac{1}{jX}O_XT$$

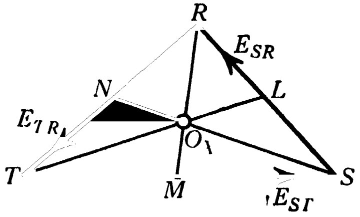


Fig. 10

from which it follows that $\mathfrak{V}_R = O_XR$: $\mathfrak{V}_S = O_XS$: $\mathfrak{V}_T = O_XT$
and since M is the mid-point of TS, therefore O_XR is $\frac{2}{3}$ MR
,, N ,, ,, ,, ,, RT ,, O_XS ,, $\frac{2}{3}$ NS
,, L ,, ,, ,, ,, SR ,, O_XT ,, $\frac{2}{3}$ LT
that is, O_X is the centre of gravity of the line pressure triangle.

When some of the turns of one phase of the stator winding of a three-phase generator are short-circuited, the e.m.f. of that phase becomes correspondingly reduced so that the line pressure triangle becomes as shown in Fig. 12, it being observed that the star pressure of the generator remains at 0 both for the symmetrical conditions of Fig. 11 and the unsymmetrical conditions of Fig. 12.

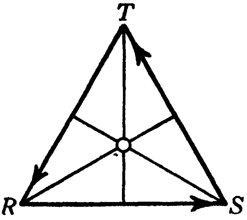


Fig. 11

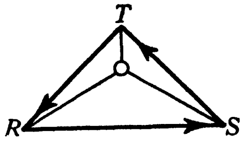


Fig. 12

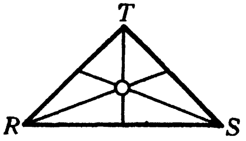


Fig. 13

If, now, three identical choking coils are connected across the lines as in Fig. 8 the pressure of the star-point of the choking coils is at the centre of gravity of the line pressure triangle as shown in Fig. 13. There will then be a pressure difference between the star-point of the reactances and the star-point of the generator. This difference of p.d. between the two star-points can be used to operate a relay giving warning that a defect in the generator winding has developed.

5.—It is required to obtain a current of variable magnitude and in phase with one of the phase pressures. Show how this may be done and derive the magnitude so obtained (i) by Thèvenin's Theorem, (ii) by means of vector diagram analysis.

ANSWER.—(i). In Fig. 14 is shown a three-phase supply and connected between the lines B and C are two equal resistances R in series. The junction D of these two resistances is connected through a variable resistance r to the line A, by means of switches s (Fig. 16.).

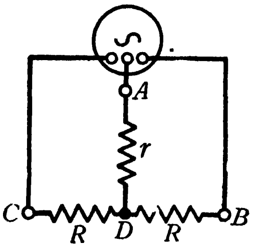


Fig. 14

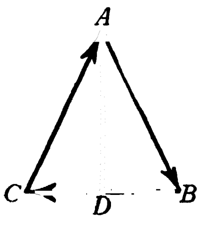


Fig. 15

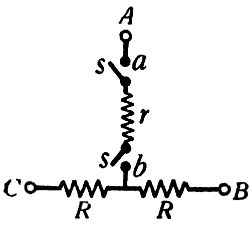


Fig. 16

When the switches are open, the pressure across the switch contacts will be (Fig. 15) $V_s = AD = \frac{\sqrt{3}}{2} V$ where V is the line voltage. Neglecting the impedance of the machine, then when the switch is open the resistance of the system between the points *a* and *b* will be $\frac{1}{2}R$ so that if these two points were to be short-circuited the current that would flow across this short-circuit would be

$$i_0 = \frac{V_S}{\frac{1}{2}R} = \frac{\sqrt{3}\,V}{R} \text{ amperes.}$$

If instead of short-circuiting *a* and *b* they are connected by the resistance *r* the current that will flow through this resistance is

$$i_r = \frac{V_S}{\frac{1}{2}R + r} = \frac{\sqrt{3}\,V}{R + 2r} \text{ amperes}$$

and this is the required solution of the problem.

(ii) The complete vector diagrammatical investigation of the system of Fig. 14 is shown in Figs. 17 to 25. The star-connected system of Fig. 14 is reproduced in Fig. 17 and the equivalent mesh connection in Fig. 18. The current in the individual phases of the mesh system are shown in Figs. 20 to 22, the respective magnitudes being

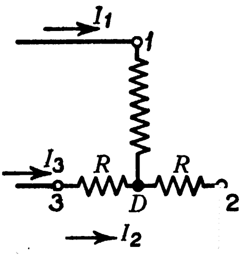


Fig. 17

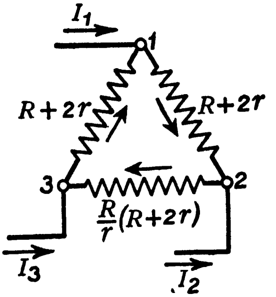


Fig. 18

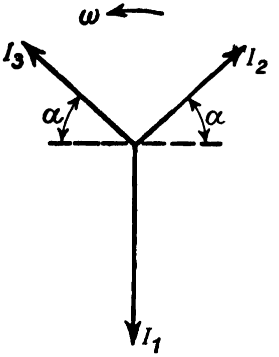


Fig. 19

$$I_{1:3} = \frac{V}{R + 2r} : I_{2.1} = \frac{V}{R + 2r} : I_{3:2} = \frac{V}{\frac{R}{r}\,(R + 2r)}$$

where *V* is the line pressure.

The respective line currents are then obtained as shown in Figs. 21 to 23, the respective magnitudes being (see also Fig. 19)

$$\text{From Fig. 21: } I_1 = 2I_{2.1} \cos 30° = \frac{\sqrt{3}\,V}{\mathrm{R} + 2r}$$

and this result is the same as obtained in (i) by Thévenin's Theorem.

Further, (see Fig. 22) $I_2 \cos \alpha = I_{3:2} + I_{2:1} \cos 60^\circ = \dfrac{V}{2R}$

and from Fig. 23: $I_3 \sin \alpha = I_{2:1} \cos 60^\circ = \dfrac{\sqrt{3}V}{2(R+2r)}$

$$I_2 = I_3$$

from which $\tan \alpha = \dfrac{\sqrt{3}\,R}{R+2r}:$

also $\cos \alpha = \sqrt{\dfrac{1}{1+\tan^2 \alpha}}$

so that $\cos \alpha \; \dfrac{R+2r}{2\sqrt{R^2+R.r+r^2}}$

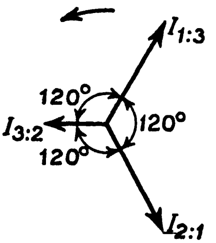


Fig. 20

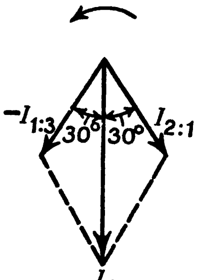


Fig. 21

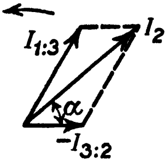


Fig. 22

and consequently, the magnitudes of I_2 and I_3 are each

$$I_2 = I_3 = \frac{V}{2R \cos \alpha} = \frac{V\sqrt{R^2+R.r+r^2}}{R(R+2r)}$$

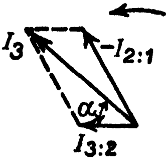


Fig. 23

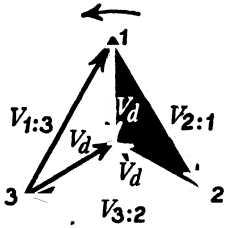


Fig. 24

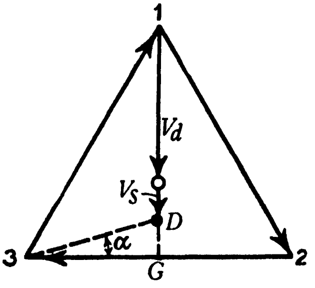


Fig. 25

If $R = r : I_2 = I_3 = \dfrac{V}{\sqrt{3}R} :$ if $r = 0 : I_2 = I_3 = \dfrac{V}{R}$

In Fig. 24 the three line pressures are shown in one vector diagram as well as the pressures to the star-point of the supply system.

As regards the pressure of the junction *D*, Fig. 17, with respect to the star-point O of the supply system, reference to Fig. 25 shows that

$$V_s = 1G - V_d - DG = \frac{\sqrt{3}}{2}V - \frac{1}{\sqrt{3}}V - DG$$

where *V* is the line pressure, that is

$$V_s = \frac{\sqrt{3}}{2}V - \frac{1}{\sqrt{3}}V - \frac{1}{2}\text{V}\tan\alpha = V\left[\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}} - \frac{\sqrt{3}\,R}{2(\text{R} + 2r)}\right]$$

$$= -\frac{V}{\sqrt{3}}\left(\frac{R-r}{R+2r}\right)$$

For $r > R$ the point *D* will lie below *O*. For $r < R$ the point D will lie above *O* in Fig. 25.

6.—One conductor of a three-phase overhead transmission system breaks and one of the free ends falls to earth; the receiver end of this line is connected to a three-phase transformer, as shown in Fig. 26. Draw the vector diagrams for the three-phase current and pressure at the consumer's terminals.

ANSWER.—The problem is illustrated in Fig. 26, from which it will be seen that the current from the supply line *R* must pass through the capacitance *C* in series with one phase of the choking coil.

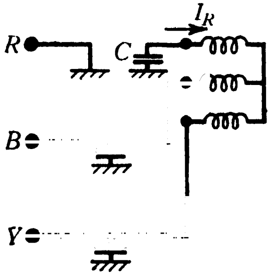


Fig. 26

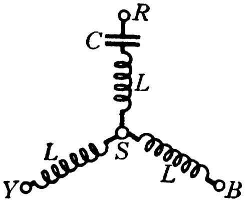


Fig. 27

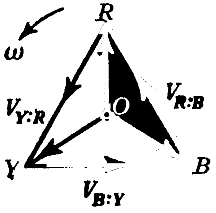


Fig. 28

Two cases will be distinguished, viz.:

(i) for which $\omega L > \frac{1}{\omega C}$

(ii) for which $\frac{1}{\omega C} > \omega L$

(i) $\omega L > \frac{1}{\omega C}$. The three-phase circuit system at the consumer end of the line is shown in Fig. 27, the star point being at *S*. Fig. 28 gives the pressure vector diagram for the supply showing the star point O of the supply system. The phase currents and pressure for the three phase system of Fig. 27 may then be found as follows :—

$$\mathfrak{V}_R=\mathfrak{I}_{RS}\,j\left(\omega L-\frac{1}{\omega C}\right):\ \mathfrak{V}_B=\mathfrak{I}_{BS}\,j\,\omega L:\ \mathfrak{V}_Y=\mathfrak{I}_{YS}\,j\,\omega L:\ \mathfrak{I}_{RS}+\mathfrak{I}_{BS}+\mathfrak{I}_{YS}=0$$

Also :

$$\mathfrak{V}_{RB}=\mathfrak{V}_R-\mathfrak{V}_B=j\left[\mathfrak{I}_{RS}\left(\omega L-\frac{1}{\omega C}\right)-\mathfrak{I}_{BS}\,\omega L\right]:$$

$$\mathfrak{V}_{YR}=\mathfrak{V}_Y-\mathfrak{V}_R=j\left[\mathfrak{I}_{YS}\,\omega L-\mathfrak{I}_{RS}\,j\left(\omega L-\frac{1}{\omega C}\right)\right]$$

$$\mathfrak{V}_{BY}=\mathfrak{V}_B-\mathfrak{V}_Y=j\,[\mathfrak{I}_{BS}\,\omega L-\mathfrak{I}_{YS}\,\omega L]$$

Substituting for $\qquad -\mathfrak{I}_{BS}=(\mathfrak{I}_{RS}+\mathfrak{I}_{YS})$,

gives, $\mathfrak{V}_{BY}=-j\,[(\mathfrak{I}_{RS}+\mathfrak{I}_{YS})\,\omega L+\mathfrak{I}_{YS}\,\omega L]=-j\,[\mathfrak{I}_{RS}\,\omega L+2\,\mathfrak{I}_{YS}\,\omega L]$

$$\mathfrak{V}_{YR}=j\left[\mathfrak{I}_{YS}\,\omega L-\mathfrak{I}_{RS}\left(\omega L-\frac{1}{\omega C}\right)\right]$$

and, eliminating $\mathfrak{I}_{YS}$ between these two equations leads to

$$\mathfrak{I}_{RS}=j\left[\frac{\mathfrak{V}_{BY}+2\,\mathfrak{V}_{YR}}{3\,\omega L-\dfrac{2}{\omega C}}\right]:\ \mathfrak{I}_{YS}=-\frac{\mathfrak{V}_{BY}+j\,\omega L\,\mathfrak{I}_{RS}}{2j\,\omega L}$$

See Figs. 29 and 30.

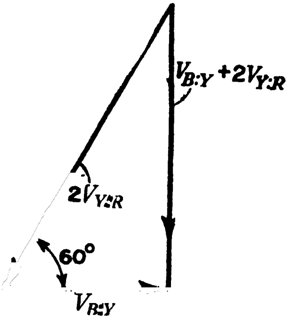


Fig. 29

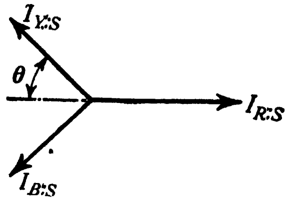


Fig. 30

Taking the following numerical data :

$$\text{Line pressure}=10kV:\ \omega L=300\Omega:\ \frac{1}{\omega C}=150\Omega$$

The vector magnitudes are:

$$|\mathfrak{J}_{\mathrm{RS}}| = \frac{\sqrt{3} \times 10{,}000}{900 - 300} = 29 \text{ amperes}:$$

$$|\mathfrak{J}_{\mathrm{YS}}| = -\,14{\cdot}5 + j\,\frac{10{,}000}{600} = -\,14{\cdot}5 + j16{\cdot}7 \text{ amperes} = 22{\cdot}1e^{j131^\circ}$$

$$|\mathfrak{B}_{\mathrm{RS}}| = \left(\omega L - \frac{1}{\omega C}\right)29 = 150 \times 29 = 4{,}350 \text{ volts}$$

$$|\mathfrak{B}_{\mathrm{BS}}| = j300 \times 22{\cdot}1 = 6{,}630 \text{ volts}$$

as shown in Fig. 31. The pressure across the inductance L in phase R is $29 \times 300 = 8{,}700$ volts, and across the inductance L in each of the phases B and Y is $22{\cdot}1 \times 300 = 6{,}630$ volts.

Case (ii) : $\dfrac{1}{\omega C} > \omega L$

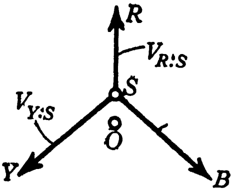


Fig. 31

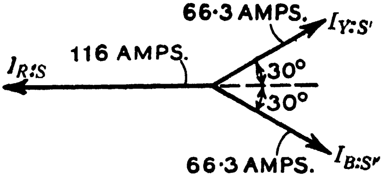


Fig. 32

Proceeding as in Case (i)

$$\mathfrak{B}_{\mathrm{BY}} = \mathfrak{B}_{\mathrm{B}} - \mathfrak{B}_{\mathrm{Y}} = j\mathfrak{J}_{\mathrm{BS}}(\omega L) - j\mathfrak{J}_{\mathrm{YS}}(\omega L):$$

$$\mathfrak{B}_{\mathrm{YR}} = \mathfrak{B}_{\mathrm{Y}} - \mathfrak{B}_{\mathrm{R}} = j\mathfrak{J}_{\mathrm{YS}}(\omega L) + j\mathfrak{J}_{\mathrm{RS}}\left(\frac{1}{\omega C} - \omega L\right)$$

$$\mathfrak{J}_{\mathrm{BS}} = -\,(\mathfrak{J}_{\mathrm{YS}} + \mathfrak{J}_{\mathrm{RS}})$$

so that

$$\mathfrak{B}_{\mathrm{BY}} = -\,2j\mathfrak{J}_{\mathrm{YS}}(\omega L) - j\mathfrak{J}_{\mathrm{RS}}(\omega L): \quad \mathfrak{B}_{\mathrm{YR}} = j\mathfrak{J}_{\mathrm{YS}}(\omega L) + j\mathfrak{J}_{\mathrm{RS}}\frac{1}{\omega C} - j\mathfrak{J}_{\mathrm{RS}}(\omega L)$$

hence,

$$\mathfrak{B}_{\mathrm{BY}} + 2\mathfrak{B}_{\mathrm{YR}} = j\mathfrak{J}_{\mathrm{RS}}\left(\frac{2}{\omega C} - 3\omega L\right)$$

and

$$\mathfrak{J}_{\mathrm{RS}} = -j\,\frac{\mathfrak{B}_{\mathrm{BY}} + 2\mathfrak{B}_{\mathrm{YR}}}{\dfrac{2}{\omega C} - 3\omega L} : \quad \mathfrak{J}_{\mathrm{YS}} = -\left[\frac{\mathfrak{B}_{\mathrm{BY}}}{2j\omega L} + \frac{1}{2}\mathfrak{J}_{\mathrm{RS}}\right]$$

Applying the following numerical data · $V = 10$ *kV line pressure:*

$$\frac{1}{\omega C} = 300 \text{ ohms} : \omega L = \frac{1}{150} \text{ ohms} :$$

$|\mathfrak{B}_{BY} + 2\mathfrak{B}_{YR}| = \sqrt{3} \times 10{,}000 = 17{,}300$ volts : $|\mathfrak{I}_{RS}|$ $= -116$ amps. : $\mathfrak{I}_{YS} = 66{\cdot}3$ amps (see Fig. 32)

The magnitudes of the pressure vectors are :

$$|\mathfrak{B}_R| = 116\left(\frac{1}{\omega C} - \omega L\right) = 17{,}500 \text{ volts}$$

$$|\mathfrak{B}_Y| = 66{\cdot}3 \times 150 = 10{,}000 \text{ volts}$$

$$|\mathfrak{B}_B| = 10{,}000 \text{ volts}$$

the vector diagrams for these pressure vectors being shown in Fig. 33.

The pressure across the inductance in phase R will be

$$I_{RS}(\omega L) = 116 \times 150 = 17{,}500 \text{ volts,}$$

and across the inductance in each of the phases B and Y will be

$$(66{\cdot}3)\omega L = 66{\cdot}3 \times 150 = 10{,}000 \text{ volts}$$

the vectors for these pressures (Fig. 33, and Fig. 34), being respectively 90° ahead of the corresponding current vectors of Fig. 32.

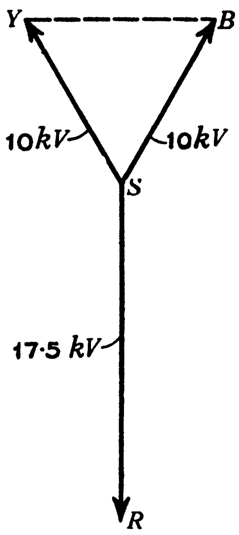


Fig. 33

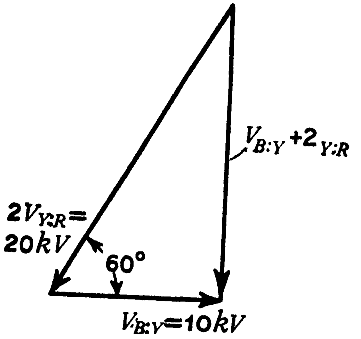


Fig. 34

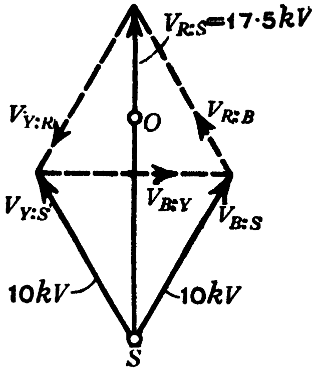


Fig. 35

The pressure between the star points S of the consumer's terminal pressure and the star point of the supply system is given by OS in Fig. 35.

Inspection of the vector diagrams for Case (ii), Figs. 32–35, disclose the remarkable and important result that the direction of phase rotation at the consumer's terminals (i.e. the transformer secondary terminals) has become reversed, and induction motors connected to these terminals will reverse their direction of rotation ; that is to say, whereas the phase sequence of the supply pressure is $\overrightarrow{R\ B\ Y}$ that of the pressure at the secondary terminals of the consumer's transformer will be $\overrightarrow{R\ Y\ B}$. Further, the very large pressure rise on one phase may result in the burning out of glow lamps and relay coils fed from this supply.

7. Show how the total reactive volt-amperes, denoted by the symbol var, taken by a balanced three-phase load may be measured by means of two wattmeters.

ANSWER.—(i) The connections for the two wattmeter method for measuring power are shown in Fig. 36 and the associated vector diagram is drawn for a balanced three-phase load. It is easily seen from this vector diagram that the power indicated by the wattmeter in line A is

$$W_A = V_{AB} I_A \cos (30 + \phi)^\circ$$

and the power indicated by the wattmeter in line C is

$$W_C = V_{CB} I_C \cos (30 - \phi)^\circ$$

so that the sum of the two wattmeter readings is

$$W = W_A + W_C = \sqrt{3} VI \cos \phi$$

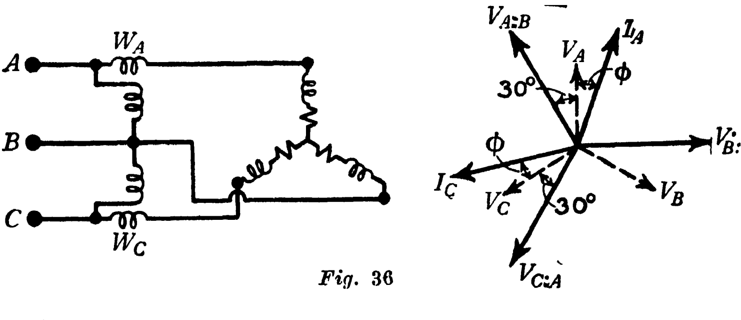


Fig. 36

where V is the line pressure and I the line current. That is to say, the sum of the two wattmeter readings gives the total three-phase power.

(ii) The connections for the two wattmeter method for measuring the reactive volt amperes (*var*) of a balanced three-phase load are shown in Fig. 37, together with the corresponding vector diagram. It will be seen from this vector diagram that the sum of the two-watt meter readings is now

$W = W_A + W_C = V_{BC}\, I_A \text{ Cos } (90° - \phi) + V_{A.B}\, I_C (\text{Cos } 90° - \phi)$ that is

$W = 2VI \sin \phi$

and is therefore proportional to the total three-phase balanced reactive volt amperes denoted by the symbol *var.*

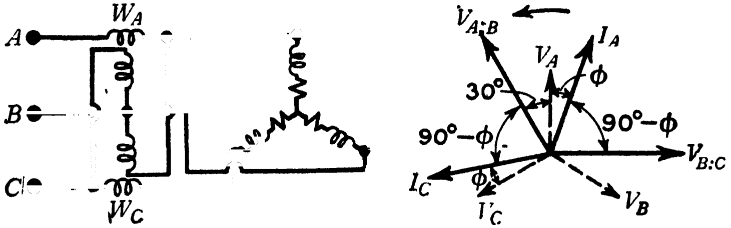


Fig. 37

Chapter XII

SOME GRAPHICAL METHODS

1.—A transformer has a 10 per cent. impedance and a short-circuit power factor of 0.4. Find the percentage pressure drop when the transformer is operating at its full rated output and (i) at unity power factor, (ii) at a power factor of 0.8 lagging.

ANSWER.—When a transformer is short-circuited a very close approximation to the actual conditions is the circuit shown in Fig. 1, in which r_1 and x_1 are respectively the resistance and reactance of the primary circuit and $r_2{}^1$ and $x_2{}^1$ the resistance and reactance respectively of the secondary winding reduced to the primary number of turns. The circuit of Fig. 1 can then be replaced by the simple series connection of Fig. 2 in which

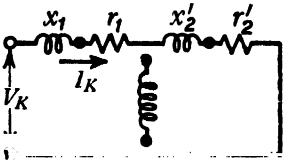


Fig. 1

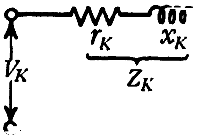


Fig. 2

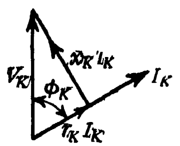


Fig. 3

$r_K = r_1 + r_2{}^1$ and is the short-circuit resistance.

$x_K = x_1 + x_2{}^1$ and is the short-circuit reactance.

$z_K = \sqrt{r_K{}^2 + x_K{}^2}$ and is the short-circuit impedance.

Also from Fig. 3 :

$$\cos \phi_K = \frac{r_K}{z_K} : \sin \phi_K = \frac{x_K}{z_K} : \tan \phi_K = \frac{x_K}{r_K} :$$

A further convenient method for expressing the relationships between these circuit constants is as follows (see Fig. 4) :

$$\cos \phi_K = \frac{I_1 r_K}{V_K} = \frac{I_1{}^2 r_K}{V_K I_1} = \frac{I_1{}^2 r_K}{V_1 I_1 \left(\frac{V_K}{V_1}\right)} = \frac{W_K\%}{V_K\%}$$

where $I_1 = I_K$ and is the normal full-load current,

V_1 is the normal supply pressure,

and $W_K\% = \frac{I_1{}^2 r_K}{V_1 I_1} 100$; $V_K\% = \frac{V_K}{V_1} 100\%$;

also $$Z_K = \frac{V_K}{I_1} = \left(\frac{\frac{V_K}{V_1}100\,V_1}{I_1\,100}\right) = \frac{V_K\%}{100}\,\frac{V_1}{I_1}$$

Again : $I_1\,r_K = V_K \cos\phi_K : I_1\,x_K = V_K \sin\phi_K$:

$$r_K = Z_K \cos\phi_K : x_K = Z_K \sin\phi_K$$

$$\mathfrak{z}_K = Z_K(\cos\phi_K + j\sin\phi_K)$$

In Fig. 5 is shown the vector diagram of Fig. 4 re-drawn with the component vectors expressed as percentages.

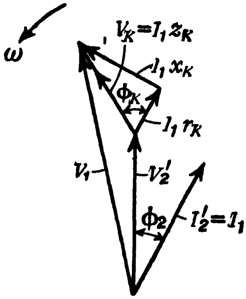


Fig. 4

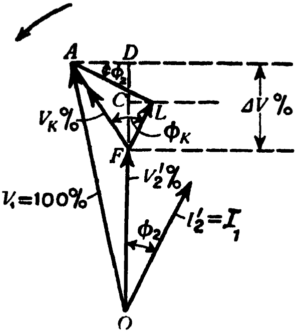


Fig. 5

then $CD = V_K\% \sin\phi_K \sin\phi : FC = V_K\% \cos\phi_K \cos\phi$

so that the percentage pressure drop at the rated normal full load is

$$\underline{\Delta V\% = CD + FC = V_K\%\,[\cos\phi\cos\phi_K + \sin\phi\sin\phi_K]}$$

Applying the following numerical data : $V_K = 10\%$: $\cos\phi_K = 0{\cdot}4$: $\sin\phi_K = 0{\cdot}92$:

(i) $\cos\phi_2 = 1 : \Delta V\% = 0{\cdot}10\,[0{\cdot}4] = 0{\cdot}04$ or $\underline{4 \text{ per cent.}}$

(ii) $\cos\phi = 0{\cdot}8 : \sin\phi = 0{\cdot}6$

$$\Delta V\% = 0{\cdot}10\,[0{\cdot}8 \times 0{\cdot}4 + 0{\cdot}6 \times 0{\cdot}92] = 0{\cdot}087 = \underline{8{\cdot}7}\,\text{per cent.}$$

2.—*Explain the general principle of operation of the synchronous motor.*

Answer.—A three-phase synchronous machine is one in which the stator is provided with a three-phase winding and the rotor comprises a field system excited by direct current. When a three-phase current of frequency f is flowing in the stator winding it will produce a magnetic field in the air-gap which will be of constant magnitude, and will rotate at the speed of n revs. per sec. where $n.p = f$, p being the number of pairs of poles for which the machine is wound. Stable operation of such a machine is only possible when the field due to the stator currents is rotating at the same speed and in the same direction as the rotor field system.

The general nature of the performance of this type of machine can be seen by reference to Figs. 6 and 7, which show portions of two concentric rings of magnet pole systems. The outer ring is built up of permanent magnets, and the inner ring is excited by means of direct current. Suppose in the first place that the two sets of magnet rings are at rest with the S poles of one ring facing the N poles of the other ring. If now the outer ring be driven by some means at a slowly increasing speed until a final steady speed of n revs. per sec. is reached, the inner ring will be carried round by the magnetic force between the two sets of poles, so that eventually, if there is no resisting torque on the axle of the inner ring, the two systems will be rotating at the same speed n with the S poles of one system facing the N poles of the other system as is shown in Fig. 6.

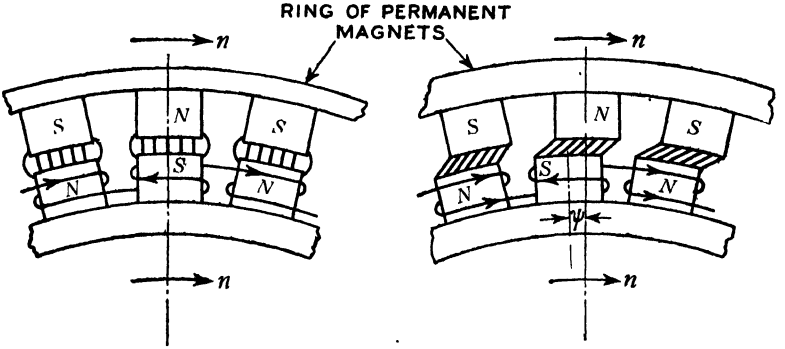


Fig. 6

Fig. 7

Now suppose that a load is coupled to the axle of the inner ring, for example, by means of a belt driving a ventilating fan. The inner ring will tend to fall back so that the two pole systems will become relatively displaced by some angle ψ, as shown in Fig. 7, and the driving torque on the inner ring will increase as the angle of displacement ψ increases, until the condition is reached that the driving torque just equals the load torque, and the two sets of rings will keep running steadily at the constant speed of n revs. per sec., but with the relative displacement of ψ electrical degrees as shown in Fig. 7. The load can then be increased, and so give rise to a corresponding increase in the angle ψ, the limit being reached when $\psi = 90$ electrical degrees. Any further increase of the load will cause the inner ring to come to a standstill and the driving will become zero. A little consideration of the foregoing discussion will show that this type of machine can only operate at " synchronous " speed, and hence it is termed a " synchronous motor " in contra-distinction to the induction motor which is an "asynchronous motor," and normally can only operate at speeds which are below the synchronous speed.

The operation of the actual synchronous motor with a three-phase stator winding, however, is somewhat more complex than the arrangement illustrated in Figs. 6 and 7, since the three-phase windings have to provide not only the driving torque but have also to produce definite armature reaction effects, as will be clear from the succeeding examples.

3.—*Draw the vector diagram for a three-phase synchronous machine which is running in parallel with a large supply system and show that the locus of the extremity of the current vector is a circle.*

ANSWER.—In Fig. 8 is shown diagrammatically by the dots and crosses in the air-gap, the direction of the three-phase currents in the stator winding, the vector diagram for these currents being also shown. If the supply pressure is V r.m.s. volts per phase, and it is assumed that this pressure is of constant magnitude and of constant frequency. The rotor is assumed to be coupled to a load so that there is a relative angular displacement of Ψ electrical degrees due to this load. The "synchronous reactance" per phase of the stator winding is X ohms and the resistance of the windings is assumed to be negligibly small. The conditions per phase can then be represented by the circuit diagram of Fig. 9 which refers to phase number 1. Since all three phases operate in precisely the same way, it is only necessary to investigate the performance of one phase.

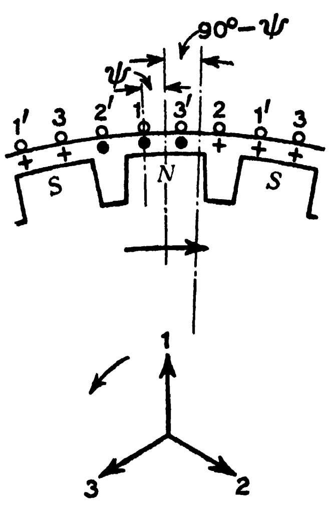


Fig. 8

The vector of the total induced e.m.f. in the machine must be equal and opposite to the applied pressure vector V_1, Fig. 10, that is, the vector OG. Assuming that the machine is running as a motor, let OA be the vector of the current I_1. There will then be only two sources of induced e.m.f. in the machine, viz., (i) the back e.m.f. due to the armature reactance X, and this back e.m.f. is of magnitude $I_1 X$ and 90° behind the vector current I_1 as shown in Fig. 10. (ii) The e.m.f. E induced by the rotor main pole system. Since these two induced e.m.f.'s must have as their vector sum the vector OG, it follows that the vector BG must give both the magnitude and phase of the induced e.m.f. E, due to the rotor field system. If the vector OF is drawn equal and parallel to the vector BG the displacement angle Θ is at once obtained (see also Fig. 11.) Further, if the excitation of the rotor field system remains constant, then as the load conditions alter the point B must move on a circle of which the centre is G and the radius is $BG = E$ the induced e.m.f. due to the rotating field system, which is assumed to be constant. The circle GBT is thus obtained as the locus of the point B.

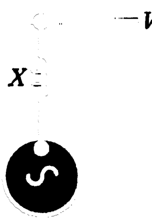


Fig. 9

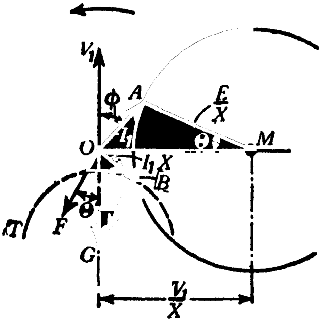


Fig. 10

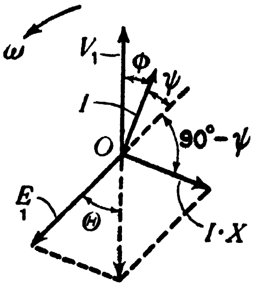


Fig. 11

Now OB is proportional to the current vector OA and is 90° behind OA, so that, since B moves on a circle, the extremity A of the current vector must also move on a circle, and a little consideration will show that the circle on which A moves has its centre at M on the abscissa axis where

$OM = \frac{V_1}{X}$, and its radius is $\frac{E}{X}$.

This then is the Circle Diagram for the synchronous machine. The upper half of this circle refers to the operation of the machine as a synchronous motor and the lower half refers to its operation as a generator. (See also "Electrician," T. F. W., 1938, April 1, page 415.)

4.—Show how the torque and power of a synchronous motor may be derived from the circle diagram.

ANSWER.—In Fig. 12 is shown the current circle diagram for the synchronous machine as derived in Example 3. Since the resistance and other losses are neglected, then for operation as a synchronous motor the electrical input will be equal to the mechanical power developed. Thus in Fig. 12 for the current vector $I = OA$ the input per phase is $V_1 I_1 \cos\phi$ watts, so that the total input for the three phases is $W = 3\, V_1 I \cos\phi$ watts.

But $I \cos\phi = AH$ and since the pressure V_1 is constant, the power developed will be directly proportional to the ordinate AH of the current vector.

If τ kg.-m. is the developed torque, then $9{\cdot}81\ \tau\ 2\pi n = 3 V_1 I \cos\phi =$ $3 V_1 \times (AH$ amperes) watts, that is $\tau = \left(\frac{3V_1}{2\pi n_0 9{\cdot}81}\right) \times (AH$ amperes) kg.-m.

where n_0 r.p.s. is the synchronous speed of the machine. The torque is thus directly proportional to the ordinate AH of the current vector.

This result can be expressed in a somewhat different way, as follows:

$$AH = AM \sin \Theta = \frac{E_1}{X} \sin \Theta$$

Where E_1 is the e.m.f. per phase to which the machine is excited.

Hence
$$\tau = \left(\frac{3V_1E_1}{2\pi n_0 9{\cdot}81\, X}\right) \sin \Theta \text{ kg.-m.}$$
that is
$$\tau = \tau_{\max} \sin \Theta \text{ kg.-m.}$$

Where $\tau_{\max}$ is the max. value of the torque which the machine can develop for the given conditions, that is, the torque corresponding to the zenith A_{ST} of the circle of Fig. 12. The important result is thus obtained that the torque is proportional to sin Θ where $\pi - \Theta$ is the angle of phase displacement between the applied p.d. V_1 and the induced e.m.f. E_1 due to the rotor field excitation (see Example 3). This relationship is shown in Fig. 13, and the crest of this sine wave marks the limit of stability of the machine for the given conditions.

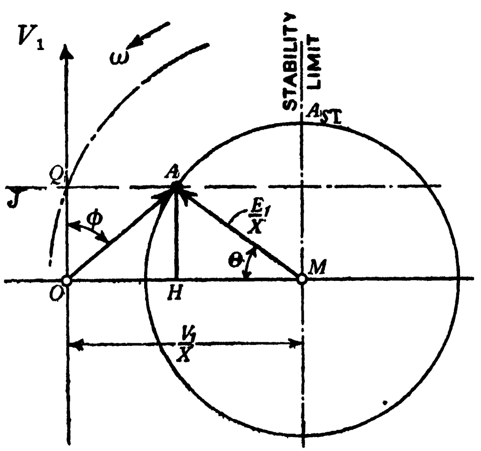


Fig. 12

Again, referring to the Circle Diagram of Fig. 12, it will be seen that since the load and the torque are each proportional to the ordinate AH, it follows that, for the given load, the extremity A of the current vector must move along the horizontal line JQA, and the position of A will be fixed by the radius of the circle, that is, by the excitation of the rotor field system. For example, if A coincides with Q, the power factor will be unity, and the excitation will then be such as to give $\frac{E_1}{X} = MQ$ amperes. If the excitation is further increased the extremity of the current vector will move to the left of the ordinate axis, and the current will lead on the applied p.d. That is to say, *if the machine is*

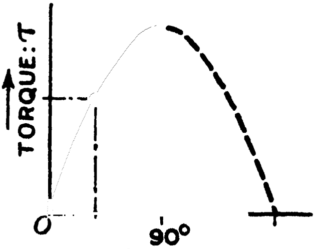


Fig. 13

"*over-excited,*" *a synchronous motor will act like a capacitance and will draw a leading current from the supply mains*—this characteristic of the synchronous motor is an extremely valuable practical feature of its operation.

5.—A synchronous motor is supplied at a constant pressure of 200 volts per phase and a frequency of 50. The reactance per phase is 8 ohms and the motor has four poles. Find the necessary excitation if the motor develops 10 horse-power at unity power factor and draw the torque/Θ curve. What excitation will be necessary if the motor is to develop 10 horse-power at a power factor of 0.9 leading? The open-circuit characteristic, that is the relationship between the induced e.m.f. E per phase and the excitation ampere-turns per pair of poles at constant speed, is given in Fig. 15.

ANSWER.—

10 H.P. = 7,460 watts input and output for the three phases.

Input per phase = 2,487 watts.

For a supply pressure of 200 volts per phase the current per phase is

$$I = \frac{2{,}487}{200} = 12{\cdot}43 \text{ amperes.}$$

The centre of the circle diagram is (Fig. 14)

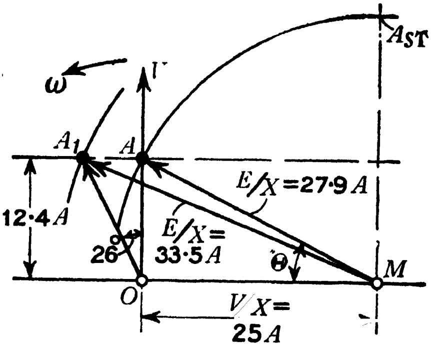


Fig. 14

$$OM = \frac{V}{X} = \frac{200}{8} = 25 \text{ amperes.}$$

The radius of the circle is then $MA = 27{\cdot}9$ amperes, so that the required value of the induced e.m.f. E in this case is given by :

$$\frac{E}{X} = 27{\cdot}9 : \text{i.e., } E = 27{\cdot}9 \times 8 = 223{\cdot}2 \text{ volts.}$$

From the characteristic curve (Fig. 15) it is seen that 5,400 excitation ampere turns per pair of poles will be necessary to induce this e.m.f.

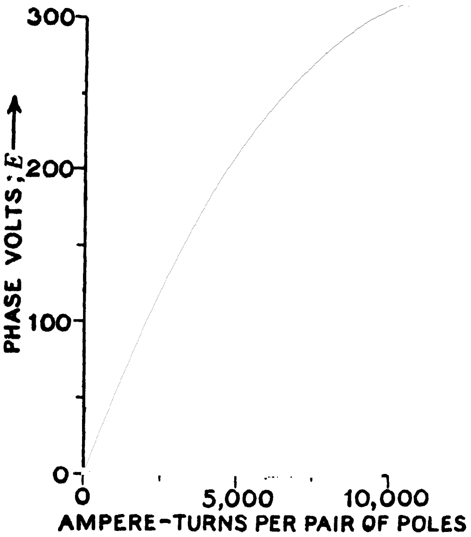


Fig. 15

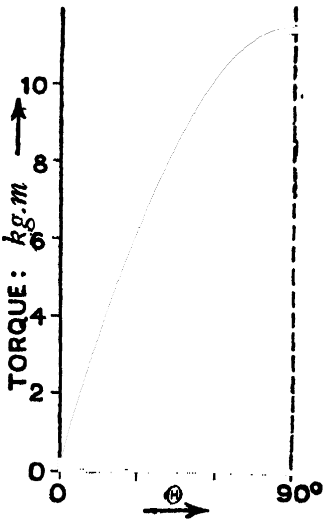


Fig 16

In order to obtain a leading power factor of 0·9 when the load of 10 H.P. is maintained the current vector OA_1 (Fig. 14) will lead by 26° C. on the applied pressure vector so that the radius of the circle must now be $OA_1 = 33{\cdot}5$ amperes, so that $E = 33{\cdot}5 \times X = 33{\cdot}5 \times 8$, that is $E = 268$ volts per phase.

Reference to the characteristic curve of Fig. 15 shows that 7,500 excitation ampere-turns per pair of poles are now necessary to develop the required e.m.f.

The torque corresponding to 10 H.P. is given by the relationship

$$\tau\,(\text{kg.-m.}) \times 9{\cdot}81 \times 2\pi n_0 = 10 \times 746$$

where $n_0 = \frac{50}{2} = 25$ r.p.s., so that $\underline{\tau = 4{\cdot}85\text{kg.-m.}}$ The torque / Θ relationship is shown in Fig. 16, for $E = 223{\cdot}2$ volts.

6.—Obtain an expression for the natural frequency of oscillation of a three-phase synchronous machine.

ANSWER.—It has been seen in Example 3 that each phase of a three-phase synchronous machine running in parallel with a large supply system of constant terminal pressure can be represented by the simple circuit system of Fig. 17, where X is the " synchronous reactance " per phase of the machine, and E the induced e.m.f. per phase due to the rotor field excitation. The vector diagram for the system of Fig. 17 is shown in Fig. 18 fort he case in which the machine is acting as a generator, that is, operating on the lower half of the circle diagram of Example 3, so that the vector of

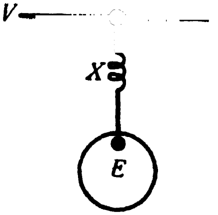


Fig. 17

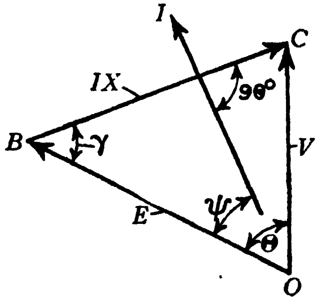


Fig. 18

induced e.m.f. E now has a component in phase with the applied pressure per phase V.

The power supplied by the machine to the supply mains is,
$W = 3EI \cos\Psi$ watts for the three phases,
that is,

$$W = \frac{3E(IX)\cos\Psi}{X} = \frac{3 \times OB \times BC}{X}\sin\gamma \text{ watts}$$

or

$$W = 3 \times \frac{(2 \text{ area of } \triangle\, OCB)}{X} = \frac{3 \times E.V \sin\Theta}{X} \text{ watts} \ldots\ldots\ldots \text{(i)}$$

Hence the torque τ with which the steam turbine is driving the machine is given by the relationship,

$$\tau \text{ (kg.-m.) } 9{\cdot}81 \times 2\pi n_0 = \frac{3EV}{X}\sin\Theta \text{ watts}$$

that is,

$$\tau = \left(\frac{3EV}{9{\cdot}81 \times 2\pi n_0 X}\right)\sin\Theta \text{ kg.-m.} = \tau_{\max}\sin\Theta \text{ kg.-m.}$$

as in Example 5 for the mechanical torque developed by a synchronous motor: this torque relationship is shown in Fig. 19. Suppose the machine is working steadily under the conditions defined by the point A_0 in Fig. 19 and assume that whilst the driving torque remains steadily constant, a sudden and transient increase in the power delivered by the machine occurs so that the condition is now defined by the point A_1, the corresponding increase of the angle $\triangle\,\Theta$ which for convenience can be written θ. The machine cannot however continue to operate at the point A_1 since the electrical power which corresponds to the point A_1 requires a torque greater than that which the steam turbine is providing. Consequently, the only way in which the extra torque can be supplied is for the machine to draw on the store of kinetic energy of its own rotating system and that to which it is coupled. This means that the machine begins to slow down and consequently the angle Θ decreases, it being observed that in this case of a *generator* the angle Θ denotes a movement in advance of the no-torque relative position whereas in the case of the synchronous *motor* of Examples 2 and 3, the angle Θ denotes a displacement of the rotor behind the no-torque relative position. As the machine slows down therefore, Θ decreases so that the condition tends towards that defined by the point A_0 in Fig. 19 which is the condition of stable operation for the given driving torque. Owing however to its inertia, the rotor overshoots the mark and swings over to the condition defined by a point, such as A_2 in Fig. 19. For values of Θ which are less than Θ_0 the driving torque exceeds the torque which corresponds to the power generated by the machine so that, having reached a lower limit A_2 the machine begins to accelerate and again overshoots the point A_0 and reaches a point in the neighbourhood of A_1. This oscillatory motion continues until it is damped out by eddy currents or by the action of specially installed damping grids arranged in the pole faces.

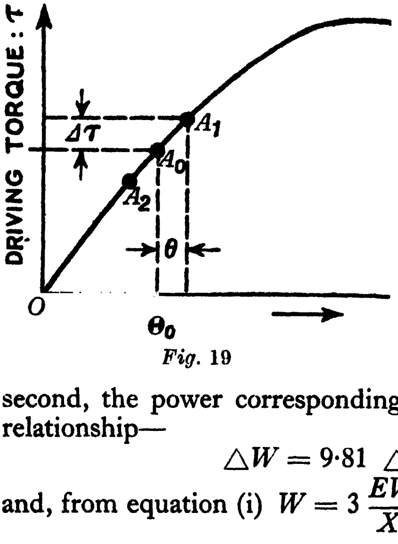


Fig. 19

If ω_0 is the synchronous angular velocity in mechanical radians per second, the power corresponding to the extra torque $\triangle\tau$ is given by the relationship—

$$\triangle W = 9{\cdot}81 \ \triangle\tau \ \omega_0 \text{ watts} \ldots\ldots\ldots\ldots \text{(ii)}$$

and, from equation (i) $W = 3\,\dfrac{EV}{X} \sin \Theta$ watts

so that, for a small angular displacement $\triangle\Theta$

$$\frac{\triangle W}{\triangle \Theta} = 3\,\frac{EV}{X} \cos \Theta$$

and,

$$\triangle W = \left[\frac{3\,EV}{X} \cos \Theta\right] \theta$$

where for convenience θ is written for $\triangle\Theta$

Equation (ii) may now be written—

$$9{\cdot}81 \ \triangle\tau \ \omega_0 = \left[\frac{3\,EV}{X} \cos \Theta\right] \theta \qquad \text{(iii)}$$

where, Θ and θ are in electrical measure.

But torque and acceleration are related by the equation:

$$\text{Torque} = \frac{J}{9{\cdot}81}\left[\begin{array}{c}\text{angular acceleration in mechanical radians/sec.}^2\\ \text{to which the torque gives rise.}\end{array}\right.$$

Where J kg.-m.^{2} units is the moment of inertia of the rotating system.

In the case now under consideration the extra torque $\triangle\tau$ gives rise to a retardation of the amount

$$-\frac{1}{p}\frac{d^2\theta}{dt^2}$$

where p is the number of pairs of poles in the machine and is introduced in order to reduce the angle θ to mechanical measure. Inserting this expression for $\triangle\tau$ in equation (iii) gives

$$\frac{d^2\theta}{dt^2} = -\left[\frac{3E\ Vp\cos\Theta}{X\,J\,\omega_o}\right]\theta$$

This denotes simple harmonic motion of oscillation frequency,

$$f_o = \frac{1}{2\pi}\sqrt{\frac{3E\ Vp\cos\Theta}{X J\omega_o}} = \frac{1}{2\pi}\sqrt{\frac{3E\ V p.\omega_o\cos\Theta}{X J\omega_o^2}}$$

or

$$f_o = \frac{1}{2\pi}\sqrt{\frac{3E\ V\,2\pi f\cos\Theta}{X\,2(K.E.)}}$$

oscillations per second, since $p.\omega_o =$ the circular electrical frequency ω, that is 2π (supply frequency f).

It is to be noted that the frequency of oscillation can be reduced by increasing the reactance X, by increasing the kinetic energy of the rotating system, *K.E.*, or by reducing the exciting current, that is reducing the e.m.f. E.

7.—Show that if there is a pair of curves such as the circles MCD and MAB in Fig. 20, and if these curves be inverted with reference to a point O, the ratio $\frac{RP}{RT}$ for an elementary area PQTR of the first pair of curves will be the same as the ratio $\frac{rp}{rt}$ for the area of inversion of the second pair of curves.

Answer.—From the definition of inversion it follows that

$$O P \times Op = k^2 = O R \times Or = O Q \times Oq = O T \times Ot \quad \text{..(i)}$$

so that,
the triangles Opr and $O R P$ are similar.
and the triangles $O R T$ and $O t R$ are similar.

Then
$$\left.\begin{aligned} \frac{R P}{r p} &= \frac{O P}{O r} = \frac{O R}{O p} \\ \frac{R T}{r t} &= \frac{O R}{O t} = \frac{O T}{O r} \end{aligned}\right\}$$
and

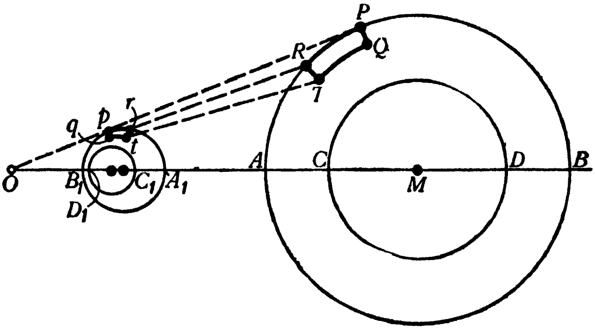


Fig. 20

so that
$$\frac{R P}{R T} = \frac{O P \times r p \times O t}{O r \times O R \times r t} = \frac{O R \times r p \times O r}{O p \times O T \times r t}$$

and consequently,

$$\left(\frac{R P}{R T}\right)^2 = \left(\frac{r p}{r t}\right)^2 \left(\frac{O P \times O t \times O R \times O r}{O r \times O R \times O p \times O T}\right) = \left(\frac{r p}{r t}\right)^2 \times \frac{O P}{O p} \times \frac{O t}{O T}$$

and since from equation (i)

$$\frac{O P}{O T} = \frac{O t}{O p}$$

and in the limit when the elements $P Q T R$ is taken indefinitely small $O P = O T$, therefore

$$\frac{R P}{RT} = \frac{r p}{r t}$$

8.—A long, straight horizontal wire of 0.8 cm. radius is supported within a cylindrical metal sheath of radius 3 cm. so that the axes are parallel and 1 cm. apart. Find the capacitance of the electrostatic field between the two conductors by the method of transforming this eccentric system into an equivalent system of a wire 0.8 cm. radius set at a definite height h above a flat metal plate parallel to the conductor.

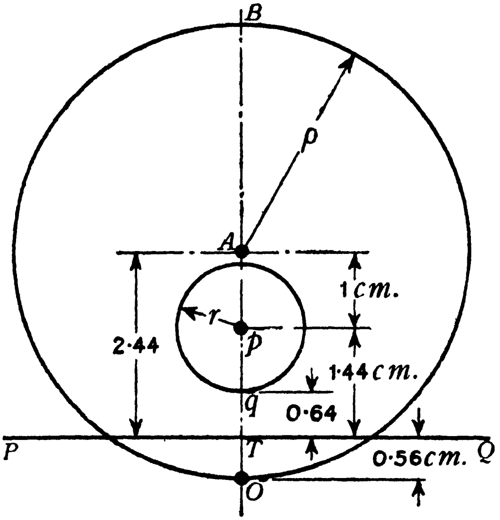


Fig. 21

ANSWER.—In Fig. 21 are shown the two eccentric circles representing the two cylindrical conductors separated by an air space. The centres of the circle are at A and p respectively, the radii being $r = 0{\cdot}8$ cm. : $\rho = 3$ cms., the distance between the centres being $Ap = 1$ cm.

This system is now to be inverted as follows: (i) The larger circle into a straight line, and this requires that the centre of inversion O lies on the circle as shown in Fig. 21. (ii) The smaller circle is to be inverted into itself, and this requires that the constant of inversion k^2 must be such that

$$Op^2 = r^2 + k^2.$$

A further relationship is that

$$OT \times OB = k^2.$$

Inserting the given numerical values, viz., $r = 0{\cdot}8$ cm. : $\rho = 3$ cms.: $Ap = 1$ cm., then

$$Op = 2 \text{ cms.},$$

so that,

$$Op^2 - r^2 = OT \times OB$$

or

$$4 - 0{\cdot}64 = OT \times 6$$

therefore,

$$OT = \frac{3{\cdot}36}{6} = 0{\cdot}56 \text{ cm.},$$

and this, therefore, fixes the position of the straight line PTQ.

From the theorem of Example 7, therefore, it follows that the capacitance of the cylinder *pq* and the plane *PTQ* is the same as the capacitance between the two eccentric cylindrical conductors *pq* and *OB*.

Referring now to Fig. 22, in which the plane *PTQ* of Fig. 21 is shown as well as the circle *pq*. From the results of expression (24), page 91, and expression (8), page 102, of "Principles" it is known that the capacitance between the cylindrical conductor *pq* and a conducting plane *PTQ* is *twice* the capacitance between the cylinder *pq* and its image p_1q_1 in the plane *PTQ*. Further, since the distance between the cylinder *pq* and its image p_1q_1 is of the same order of magnitude as the radii of these cylinders, the exact formula derived in Example 14, Chapter IV, must be used.

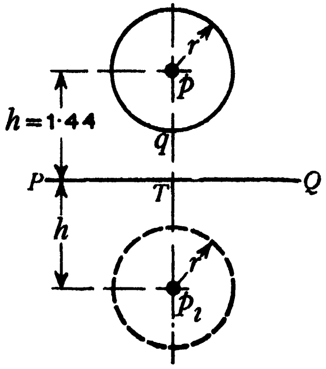


Fig. 22

Hence: the capacitance of the space between the two eccentric cylindrical conductors of Fig. 21 is :—

$$C = 2\left\{\frac{1}{51{\cdot}4\log_{10}\left[\dfrac{\sqrt{(2h)^2-(2r)^2}+2h}{2r}\right]}\right\}\mu F/\text{mile}$$

$$= \frac{1}{25{\cdot}7\ log_{10}[3{\cdot}3]}\ \mu F/\text{mile}$$

$$= 0.075\mu F/\text{mile}$$

or $\underline{C = 0.047\ \mu F/\text{km}}$.

9.—Find the capacitance of the system of two eccentric cylindrical conductors defined in Example 8 by the method of transforming the eccentric system into an equivalent system of two concentric conductors.

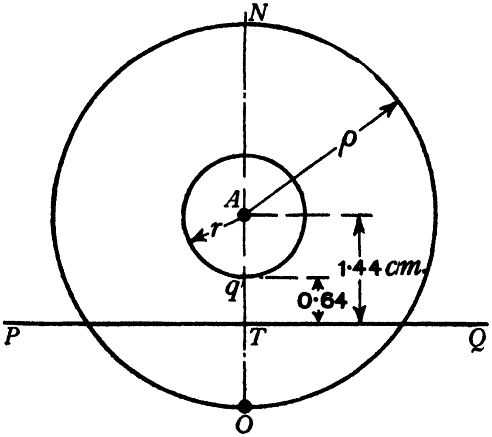


Fig. 23

ANSWER.—In Fig. 23 the circle *Aq* and the plane *PTQ* are shown in their correct relative position as has been found already in the solution of Example 8, so that the first step in the present procedure has been to find the position of the line *PTQ* which is, from Example 8, known to be distant 1·44 cms. from the centre of the circle *Aq*.

The next step is to invert the system of circle Aq and line PTQ into a system of two concentric circles, viz.: such that the circle Aq inverts into *itself* and the line PTQ inverts into a circle which is concentric with the circle Aq. As regards this latter requirement it is clear that the centre of inversion must lie on the circle which is obtained by inverting the line PTQ. The conditions will therefore be as shown in Fig. 23 and all that remains is to find the radius ρ of the new circle, and this is easily done by means of the relationship $OA^2 = r^2 + k^2 : OT \times ON = k^2$

$$\therefore \quad (OT + AT)^2 - r^2 = OT \times 2\rho$$

$$(OT + 1{\cdot}44)^2 - 0{\cdot}8^2 = 2 \times OT\,(OT + 1{\cdot}44)$$

$$OT^2 + 2{\cdot}88(OT) + 2{\cdot}07 - 0{\cdot}64 = 2(OT)^2 + 2{\cdot}88(OT)$$

or $$(OT)^2 = 1{\cdot}43 : OT = 1{\cdot}2$$

and $$\rho = 1{\cdot}2 + 1{\cdot}44 = 2{\cdot}64 \text{ cms.}$$

The required capacitance therefore is the capacitance between two concentric cylinders of radii respectively $r = 0{\cdot}8$ cm.: $\rho = 2{\cdot}64$ cms., so that

$$C = \frac{1}{25{\cdot}7 \log_{10}\left(\frac{2{\cdot}64}{0{\cdot}8}\right)} = \frac{1}{25{\cdot}7 \log_{10} 3{\cdot}3}\ \mu F/\text{mile}$$

$$= \frac{1}{25{\cdot}7 \times 0{\cdot}518} = \frac{1}{13{\cdot}3} = 0{\cdot}075\ \mu F/\text{mile}$$

or $$\underline{C = 0{\cdot}047\ \mu F/\text{km.}}$$

as already found in Example 8.

10.—*Apply the results obtained for the capacitance between two eccentric cylindrical conductors to determine the magnetic reluctance of the space between a cylindrical iron rod which is supported in a cylindrical tunnelled hole in a laminated iron block, the axes of the rod and the tunnelled hole being parallel and at a definite distance apart.*

ANSWER.—In Fig. 24 the smaller circle represents the iron rod and the larger circle the tunnelled hole, the amount of eccentricity being given by the distance y.

The *magnetic conductivity* $\left(\text{i.e. } \frac{1}{\text{reluctance}}\right)$ is given by an expression of the same form as that for the electrostatic capacity of the field between the two cylindrical conductors which have been considered in Examples 8 and 9.

Suppose in the first place that the eccentricity y is zero, so that the cylindrical rod and the tunnelled hole are co-axial as shown in Fig. 25, then the magnetic reluctance can be expressed as follows. Consider a co-axial band of the air-space of radius x cm. and of radial breadth δx cm. as shown in Fig. 25, then the magnetic reluctance of the band will be

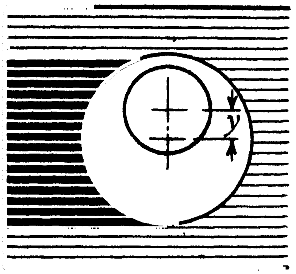


Fig. 24

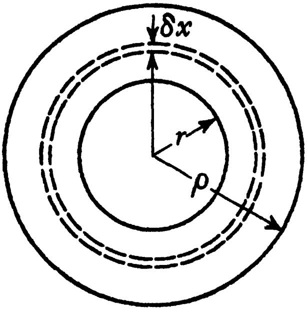


Fig. 25

$$\frac{\delta x}{2\pi x} \text{ units per cm. axial length.}$$

so that the reluctance of the whole air-space between the two co-axial surfaces will be.

$$\int_r^\rho \frac{dx}{2\pi x} = \frac{1}{2\pi}\log_e\frac{\rho}{r} \text{ per cm. per axial length}$$

and this expression is precisely similar in form to the reciprocal of the electrostatic capacitance of a concentric cable (see page 103, " Principles "). By means of this identity of form of the expressions for magnetic conductivity and the electrostatic capacitance of concentric cylindrical surfaces, viz.,

$$\frac{1}{2\pi}\log_e\frac{\rho}{r} = \frac{1}{4\pi\left\{\begin{array}{c}\text{Electrostatic capacitance in c.g.s. units}\\ \text{per cm. length of concentric cable}\end{array}\right\}} \quad \ldots.(i)$$

it becomes at once possible to write down the expression for the magnetic reluctance when the cylindrical rod is set eccentrically with the tunnelled hole as shown in Fig. 24.

From the results of the theorems of Examples 8 and 9 of this Test Paper, it is known that the capacitance between the two eccentric cylindrical surfaces Cq and MB of Fig. 26, is the same as that between the cylindrical surface Cq and the plane PQ when the following conditions are satisfied, viz.,

(1) The circle Cq is inverted into itself.
(2) The circle MB is inverted into the straight line PQ.

These conditions will be fulfilled if O is the centre of inversion where,

$$(OC)^2 - r^2 = k^2 = (OT) \times (OB)$$

that is, if,

$$(\rho - y)^2 = r^2 + 2\rho(\rho - \Delta - y)$$

or

$$\Delta = (\rho - y)\left[1 - \frac{\rho - y}{2\rho}\right] + \frac{r^2}{2\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

In Fig. 27 the circle Cq and the plane PQ of Fig. 26 are reproduced, and from what has been said in the foregoing it will be seen that, the capacitance of the two eccentric surfaces of Fig. 26 will be the same as the capacitance of the air-space between the cylindrical surface Cq and the plane PQ of Fig. 27. But this capacitance is twice that between the cylindrical surface Cq and its image $C'q'$ in the plane PQ. That is to say, making use of the result given in Example 14 of Test Paper, Chapter IV, it will be seen that the capacitance between the two eccentric surfaces of Fig. 26 is

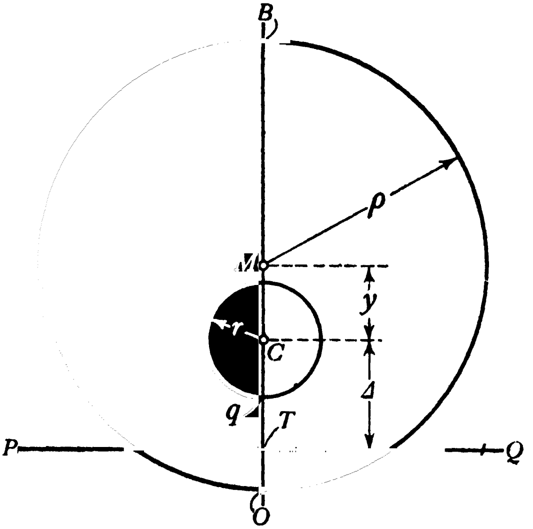


Fig. 26

$$C = 2\left[\frac{1}{4\ \log_e \left[\dfrac{\sqrt{\Delta^2 - r^2} + \Delta}{r}\right]}\right] \begin{array}{l}\text{electrostatic units}\\ \text{per cm. length.} \ \ldots\ldots\ \text{(iii)}\end{array}$$

and consequently, from expression (i), the magnetic reluctance between the two eccentric surfaces of Fig. 26 is

$$\frac{1}{4\pi}\left\{2 \log_e \left[\frac{\sqrt{\Delta^2 - r^2} + \Delta}{r}\right]\right\} \text{units per cm. length} \ \ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

Numerical example $_2$

$\rho = 1\frac{9}{16}'' = 3.96$ cm. : $r = 1\frac{1}{16}'' = 2.70$ cm.

eccentricity $y = \frac{3}{16}'' = 0.476$ cm.

so that, from (ii)

$$\Delta = (3.96 - 0.476)\left[1 - \frac{3.96 - 0.476}{2 \times 3.96}\right] + \frac{2.7^2}{2 \times 3.96}.$$

that is,

$$\Delta = 2.86 \text{ cm.}$$

From (iv) the required magnetic reluctance is then

$$\frac{1}{2\pi}\log_e\left[\frac{\sqrt{2.86^2 - 2.7^2} + 2.86}{2.7}\right]$$

$$= \frac{2.3}{2\pi}\log_{10}\frac{3.81}{2.7} = \underline{0.055 \text{ units per cm. length.}}$$

In Fig. 28 the values so obtained for the magnetic reluctance *per mm* axial length are shown as a function of the eccentricity y.

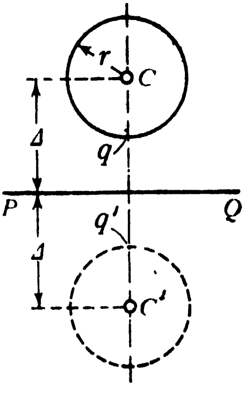


Fig. 27

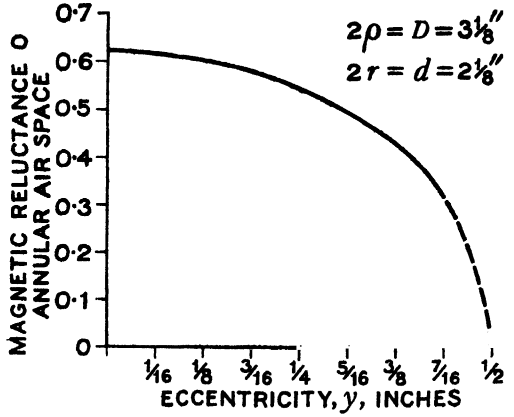


Fig. 28

11.—Obtain the expression for the magnetic reluctance of the air-space between the two eccentric cylindrical surfaces of Fig. 24, Example 10, by inverting the two eccentric surfaces into two equivalent co-axial surfaces.

For the solution of this problem it is necessary to invert the cylindrical surface Cq and the plane surface PQ of Fig. 26, Example 10, into a system of co-axial cylindrical surfaces.

Referring now to Fig. 29 the circle Cq and the plane PQ are identically the same as those in Fig. 26 and 27, and it is now necessary to find the centre of inversion O* such that,

(i) The circle Cq inverts into itself.

(ii) The plane PQ inverts into the circle CN which is co-axial with the circle Cq.
That is to say, the requirements will be satisfied if the radius R of the circle $O^{*}N$, Fig. 29 is found. This is easily obtained as follows:

$$(O^{*}C)^{2} = r^{2} + k^{2}$$

$$(O^{*}T) \times 2R = k^{2}$$

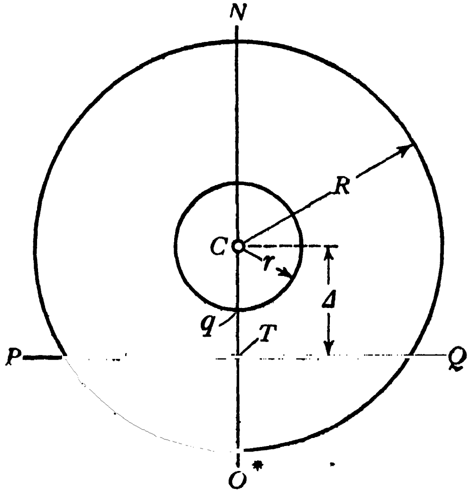


Fig. 29

so that

$$R^{2} = r^{2} + 2R\,(R - \triangle) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

where $\triangle$ has the value already obtained in Example 10, expression (ii), viz.,

$$\triangle = (\rho - y)\left[1 - \frac{\rho - y}{2\rho}\right] + \frac{r^{2}}{2\rho}$$

Taking the numerical data already given in Example 10, viz.,

$$\rho = 1\tfrac{9}{16}'' = 3.96 \text{ cm.}: \quad r = 1\tfrac{1}{16}'' = 2.70 \text{ cm.}$$

$$\text{eccentricity } y = \tfrac{3}{16}'' = 0.476 \text{ cm.}$$

Then,

$$\triangle = 2.86 \text{ cm.}$$

and,

$$R^{2} = 2{\cdot}7^{2} + 2R\,(R - 2{\cdot}86)$$

or,

$$R^{2} - 5{\cdot}72R + 7{\cdot}28 = 0$$

from which it is found that,

$$R = 3{\cdot}84 \text{ cm.}$$

The capacitance between two co-axial cylindrical surfaces of radii $R = 3{\cdot}84$ cm. and $r = 2{\cdot}70$ cm.
is

$$C = \frac{1}{2 \log_e \frac{R}{r}} \text{ elec. st. units per cm. length}$$

that is

$$C = \frac{1}{2 \log_e \frac{3{\cdot}84}{2{\cdot}70}} = \frac{1}{0{\cdot}69}$$

so that the required magnetic reluctance is given by expression (i), Example 10, viz.,

$$\frac{1}{4\pi} \times \frac{1}{C} = \frac{1}{4\pi} \times 0{\cdot}69$$

$$= \underline{0{\cdot}055 \text{ units per cm. length}}$$

and this agrees with the value found in Example 10.

In Fig. 30 is shown the relationship between the magnetic reluctance per cm. length and the eccentricity y for the following dimensions of the eccentric magnetic system:

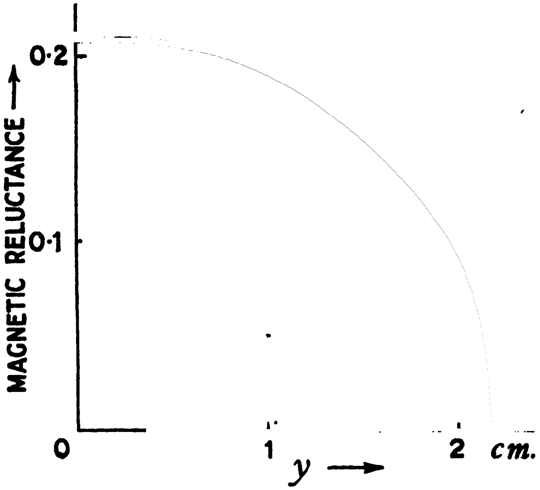


Fig. 30

$\rho = 3$ cm.: $r = 0{\cdot}8$ cm:

The curve of Fig. 30 has been derived by the method described in this present Example 11 (see T.F.W., Journal of Inst. of Elec. Engineers, 1929, page 906).

12.—By means of the method of symmetrical component analysis derive an expression for the current which will flow when one phase of a generator is short-circuited to the star point and the other two phases are open.

ANSWER.—It is to be noted that the short-circuit current $\mathfrak{I}_a$ of Fig. 31,

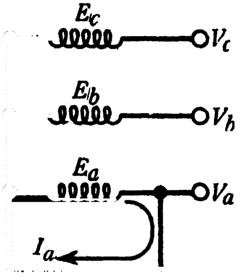


Fig. 31

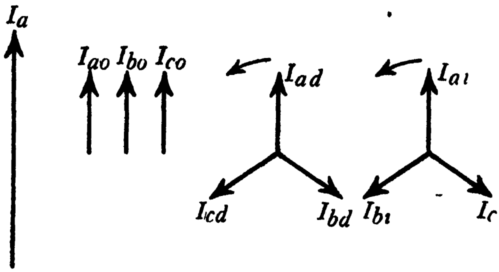


Fig. 32

is a single-phase current, that is to say, it is an unsymmetrical three-phase current for which $\mathfrak{I}_b = 0$, and $\mathfrak{I}_c = 0$, as shown in Fig. 32.

The three symmetrical components are respectively given by the following three simultaneous equations :

$$\begin{aligned}
\mathfrak{I}_{ao} &= \tfrac{1}{3}\,(\mathfrak{I}_a + \mathfrak{I}_b + \mathfrak{I}_c) \\
\mathfrak{I}_{ad} &= \tfrac{1}{3}\,(\mathfrak{I}_a + a\,\mathfrak{I}_b + a^2\,\mathfrak{I}_c) \\
\mathfrak{I}_{a1} &= \tfrac{1}{3}\,(\mathfrak{I}_a + a^2\,\mathfrak{I}_b + a\,\mathfrak{I}_c)
\end{aligned}$$

Where $a = e^{j\frac{2\pi}{3}}$: $a^2 = e^{j\frac{4\pi}{3}}$: since however, $\mathfrak{I}_b = 0$ and $\mathfrak{I}_c = 0$ it follows at once that the symmetrical components are :

$$\mathfrak{I}_{ao} = \mathfrak{I}_{ad} = \mathfrak{I}_{a1} = \tfrac{1}{3}\,\mathfrak{I}_a$$

as shown in Fig. 32.

The symmetrical three-phase induced e.m.f.'s due to the field excitation of the generator is defined by the vector diagram of Fig. 32 and the following three simultaneous equations relate the symmetrical components to the induced e.m.f. components, viz. :

$$\left.\begin{aligned}
\mathfrak{E}_a &= j\mathfrak{I}_{ad}X_1 + j\mathfrak{I}_{a1}X_2 + j\mathfrak{I}_{ao}X_o \\
\mathfrak{E}_b &= \mathfrak{V}_b + j\mathfrak{I}_{ad}X_1 + j\mathfrak{I}_{b1}X_2 + j\mathfrak{I}_{bo}X_o \\
\mathfrak{E}_c &= \mathfrak{V}_c + j\mathfrak{I}_{cd}X_1 + j\mathfrak{I}_{c1}X_2 + j\mathfrak{I}_{co}X_o
\end{aligned}\right\} . \qquad (1)$$

where X_1 is the reactance per phase for the " direct " rotating three-phase current system.

X_2 is the reactance per phase for the " inverse " rotating three-phase current system.

X_o is the reactance per phase for the " null current " system.

The following relationships also hold for the short-circuited phase *a* :

$$\left.\begin{aligned} \mathfrak{E}_a &= j\,\mathfrak{J}_{ad}\,X_1 + \mathfrak{B}_{ad} \\ 0 &= j\,\mathfrak{J}_{ai}\,X_2 + \mathfrak{B}_{ai} \\ 0 &= j\,\mathfrak{J}_{ao}\,X_o + \mathfrak{B}_{ao} \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

since the induced e.m.f. system $\mathfrak{E}$, of course, is a purely direct rotating system. Further,

$$\mathfrak{B}_a = 0 = \mathfrak{B}_{ad} + \mathfrak{B}_{ai} + \mathfrak{B}_{ao}$$

so that, adding together the three-equation (2) gives

$$\mathfrak{E}_a = j\,(\mathfrak{J}_{ad}\,X_1 + \mathfrak{J}_{ai}\,X_2 + \mathfrak{J}_{ao}\,X_o)$$

and since $\qquad \mathfrak{J}_{ao} = \mathfrak{J}_{ad} = \mathfrak{J}_{ai}$ (Fig. 31),

therefore $\qquad j\,\mathfrak{J}_{ad} = \dfrac{\mathfrak{E}_a}{X_1 + X_2 + X_o}$

and $\quad \mathfrak{B}_{ao} = -j\,\mathfrak{J}_{ao}\,X_o : \mathfrak{B}_{ai} = -j\,\mathfrak{J}_{ai}\,X_2 : \mathfrak{B}_{ad} = \mathfrak{E}_a - j\,\mathfrak{J}_{ad}\,X_1$ (see Fig. 34).

In an actual short-circuit test of a three-phase machine, as shown in Fig. 31, the induced e.m.f. per phase as obtained from the open-circuit characteristic was 105 volts, whilst the measured values of the respective reactances were :

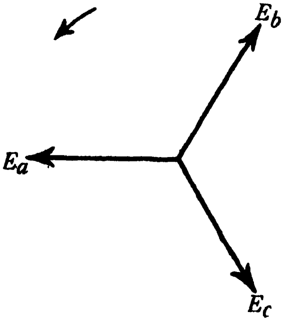


Fig. 33

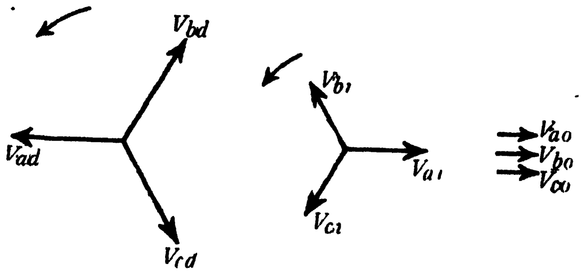


Fig. 34

$$X_1 = 7{\cdot}15 \text{ ohms} : X_2 = 1 \text{ ohm} : X_o = 0{\cdot}25 \text{ ohm}.$$

The magnitudes of the symmetrical components are consequently given by

$$I_{ad} = \frac{105}{7{\cdot}15 + 1 + 0{\cdot}25} = 12{\cdot}5 \text{ amperes} = I_{ai} = I_{ao}$$

and the magnitudes of the respective terminal pressures are then

$$V_{ao} = I_{ao} X_o = 12{\cdot}5 \times 0{\cdot}25 = 3{\cdot}12 \text{ volts}$$

$$V_{ad} = E_a - I_{ad} X_1 = 105 - 12{\cdot}5 \times 7{\cdot}15 = 15{\cdot}6 \text{ volts}$$

$$V_{ai} = I_{ai} X_2 = 12{\cdot}5 \times 1 = 12{\cdot}5 \text{ volts}$$

so that (see Fig. 31) the actual short-circuit current which will flow in phase *a* will be

$$I_a = 3\, I_{ao} = 3 \times 12{\cdot}5 = 37{\cdot}5 \text{ amperes.}$$

Actually, the value measured on test was 37·0 amperes.

For further examples of short-circuit analysis see " Electrical Review," July 9th, 1943 (T. F. W.).

Chapter XIII

NON-SINUSOIDAL WAVE FORMS : HARMONIC ANALYSIS EFFECT OF WAVE FORM ON ELECTRICAL MEASUREMENTS

1.—The primary winding of a single-phase transformer is connected to a pressure wave of sinusoidal form and frequency f hz. Neglecting the resistance of the winding, obtain an expression for the induced back e.m.f.

ANSWER.—The induced back e.m.f. is given by the expression $e = -\frac{d}{dt}(\phi.w)10^{-8}$ volts where $\phi.w$ is the number of flux-linkages of the coil at any moment t. Since the applied p.d. is sinusoidal, this induced e.m.f. must also be sinusoidal. Further, in order that the rate of change of $\phi.w$ shall be sinusoidal, the flux ϕ must itself be a sine function of the time.

If Φ is the peak value of the flux wave, then the flux-linkages will vary from $\Phi.w$ to zero in one-quarter cycle, i.e., in $\frac{1}{4f}$ sec., so that the mean rate of change will be

$$\frac{\Phi.w}{\frac{1}{4f}}\ 10^{-8} \text{ volts}$$

and consequently, the mean value of the induced e.m.f. will be,

$$E_{av} = 4f\ \Phi.w\ 10^{-8} \text{ volts}$$

Since the wave form is sinusoidal, the peak value of the e.m.f. will be,

$$E_m = \frac{\pi}{2} E_{av}$$

so that the r.m.s. value will be

$$E = \frac{\pi}{2\sqrt{2}} E_{av}$$

that is $$E = \frac{4\pi}{2\sqrt{2}} f\ \Phi.w10^{-8} = 4{\cdot}44 f\ \Phi\ w10^{-8} \text{ r.m.s. volts.}$$

If, for example,

$$f = 50 \text{ h}_z\text{.,}\ \Phi = 9 \times 10^6 : w = 500$$

then.

$$E = 4{\cdot}44 \times 50 \times 9 \times 10^6 \times 500 \times 10^{-8} = 10{,}000 \text{ r.m.s. volts}$$

2.—In Fig. 1a is shown a sine wave form of flux of peak value 9 × 10^6 c.g.s. lines, as a function of the circular frequency $\omega = 2\pi f$, and in Fig. 1b is shown the relationship between the magnetising current and the flux. Obtain the wave form of the magnetising current.

ANSWER.—The peak A of the flux wave in Fig. 1*a* corresponds to the the point A in Fig. 1*b* so that the peak value of the magnetising current is 12 amperes. Similarly, for the point P on the flux wave of Fig. 1*a* the corresponding point P on the current wave shows that the magnetising current for this flux is 1·5 amperes. In this way the whole wave of magnetising current can be obtained as shown by the full-line curve in Fig. 2

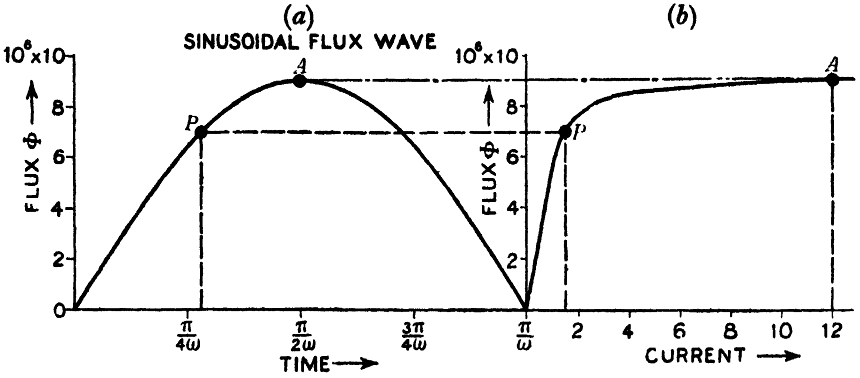


Fig. 1

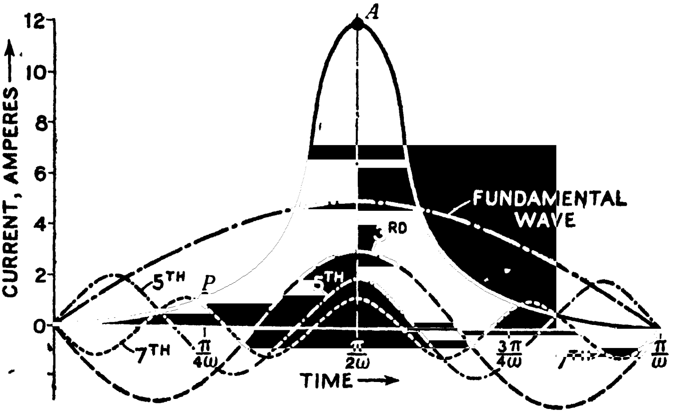


Fig. 2

where the points P and A correspond to the points P and A of Figs. 1*a* and Fig. 1*b*. This wave is typical of the distortion which is produced when the core of the transformer is working at flux density, which is high up on the saturation curve.

3.—*Analyse, as far as the seventh harmonic, the current wave form obtained in Example 2.*

TABLE

Ordinates are taken spaced at every 5°, thus giving 36 ordinates for the complete half wave.

There are no cosine components in this analysis The factor $\frac{2}{m}$ is thus $= \left(\frac{1}{18}\right)$

Current Ordinate No	Amperes	$\theta°$	Sin θ	Fundamental Sine Wave	Third Harmonic Sine Wave	Fifth Harmonic Sine Wave	Seventh Harmonic Sine Wave
1	0·1	5°	0·087	i_1 sin 5° = 0·009	i_1 sin 15° = 0·026	i_1 sin 25° = 0·04	i_1 sin 35° = 0·05
2	0·2	10°	0·174	i_2 sin 10° = 0·053	i_2 sin 30° = 0·188	i_2 sin 50° = 0·13	i_2 sin 70° = 0·19
3	0·3	15°	0·259	i_3 sin 15° = 0·078	i_3 sin 45° = 0·212	i_3 sin 75° = 0·25	i_3 sin 105° = 0.25
4	0·4	20°	0·342	i_4 sin 20° = 0·137	i_4 sin 60° = 0·348	i_4 sin 100° = 0·40	i_4 sin 140° = 0·26
5	0·5	25°	0·423	i_5 sin 25° = 0·211	i_5 sin 75° = 0·483	i_5 sin 125° = 0·41	i_5 sin 175° = 0.04
6	0·65	30°	0·500	i_6 sin 30° = 0·325	i_6 sin 90° = 0·650	i_6 sin 150° = 0·32	i_6 sin 210° = − 0·35
7	0·80	35°	0·574	i_7 sin 35° = 0·460	i_7 sin 105° = 0·766	i_7 sin 175° = 0·05	i_7 sin 245° = − 0·71
8	0·90	40°	0·643	i_8 sin 40° = 0·580	i_8 sin 120° = 0·780	i_8 sin 200° = − 0·36	i_8 sin 280° = − 0·88
9	1·10	45°	0·707	i_9 sin 45° = 0·780	i_9 sin 135° = 0·770	i_9 sin 225° = − 0·71	i_9 sin 315° = − 0·71
10	1·4	50°	0·766	i_{10} sin 50° = 1·07	i_{10} sin 150° = 0·70	i_{10} sin 250° = − 1·30	i_{10} sin 350° = − 0·24
11	1·7	55°	0·819	i_{11} sin 55° = 1·39	i_{11} sin 165° = 0·446	i_{11} sin 275° = − 1·70	i_{11} sin 385° = + 0·70
12	2·2	60°	0·866	i_{12} sin 60° = 1·90	i_{12} sin 180° = 0	i_{12} sin 300° = − 1·90	i_{12} sin 420° = 1.90
13	2·9	65°	0·906	i_{13} sin 65° = 2·63	i_{13} sin 195° = − 0·76	i_{13} sin 325° = − 1·80	i_{13} sin 455° = 2·80
14	4·1	70°	0·940	i_{14} sin 70° = 3·84	i_{14} sin 210° = − 2·00	i_{14} sin 350° = − 0·70	i_{14} sin 490° = 3·10
15	6·2	75°	0·966	i_{15} sin 75° = 6·00	i_{15} sin 225° = − 4·40	i_{15} sin 375° = 1·60	i_{15} sin 525° = 1·64
16	10·0	80°	0·985	i_{16} sin 80° = 9·85	i_{16} sin 240° = − 8·66	i_{16} sin 400° = 6·43	i_{16} sin 560° = − 3·40
17	11·5	85°	0·996	i_{17} sin 85° = 11·10	i_{17} sin 255° = − 11·20	i_{17} sin 425° = 10·40	i_{17} sin 595° = − 9·30
18	12·0	90°	1·000	i_{18} sin 90° = 12·00	i_{18} sin 270° = − 12·00	i_{18} sin 450° = 12·00	i_{18} sin 630° = − 12·00
				TOTAL 52·395	TOTAL − 33·68	TOTAL 22·56	TOTAL − 16·92

It will be seen that the data for the rest of column 5 of this Table for the ordinate Nos 19 to 36 inclusive will be the same as for ordinates Nos 17 to 0 inclusive, so that for the complete 36 ordinate the total sum of column 5 will be the same as that obtained by taking the sum of Nos 1 to 17 adding half of No 18 and multiplying the result by two The same result is obtained by taking the sum of Nos 1 to 18, subtracting one-half of No 18 and multiplying the result by two

Total for complete half cycle is

$$2(52·395 - 6) = 2 \times 46·4 = 92·8$$

$$\therefore A_1 = \frac{2}{m}(92·8) = \frac{92·8}{18}$$

$$\text{i.e., } \underline{A_1} = \underline{5·15}$$

Total for complete half cycle of the fundamental wave is

$$2 \times -(33·68 - 6) = -55·36$$

$$\underline{A_3} = -\frac{55·36}{18} = \underline{-3·07}$$

Total for complete half cycle of the fundamental wave is

$$2(22·56 - 6) = 33·16$$

$$\underline{A_5} = \frac{33·16}{18} = \underline{1·84}$$

Total for complete half cycle of the fundamental wave is .

$$2 \times -(16·92 - 6) = -21·84$$

$$\underline{A_7} = -\frac{21·84}{18} = \underline{-1·21}$$

The given distorted wave of magnetising current, analysed to the seventh harmonic is

$$\underline{i = 5·15 \sin \omega t - 3·07 \sin 3\,\omega t + 1·84 \sin 5\,\omega t - 1·21 \sin 7\,\omega t}$$

4.—Currents of different frequencies are flowing in a circuit as follows :

Frequency : hertz	***0*** ***(i.e.,d.c.)***	***50***	***150***	***750***
Current in r.m.s. amperes	***10***	***10***	***5***	***3***

Find the r.m.s. value of the current indicated by an a.c. ammeter in series with the circuit.

Answer.—The resultant r.m.s. value of the current is given by the expression :

$$I = \sqrt{I_1^2 + I_2^2 + I_3^2 + \ldots}$$

so that I can be found by the graphical construction of Fig. 3, a D.C. current being regarded as an A.C. current of zero frequency. The resultant current is therefore given by the vector $O\,D$, viz., $I = 15{\cdot}3$ r.m.s. amperes.

A similar construction can, of course, be applied to obtain the resultant r.m.s. pressure of a circuit in which a number of pressures of different frequencies are connected in series.

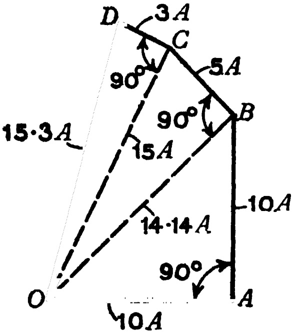


Fig. 3

5.—Since a third harmonic cannot flow in a three-phase supply system with an insulated star point, state what means are available for enabling the requisite third harmonic of the current wave to flow in the transformer winding.

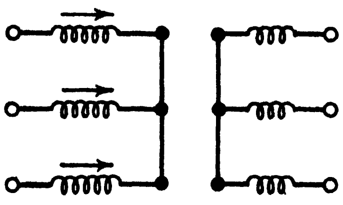


Fig. 4

Answer.—If both the primary and the secondary windings of the transformer are star-connected, triple harmonic currents cannot flow in either winding since the individual currents in each limb will all be in phase and consequently there will be no return path for their circuit (see Fig. 4). If, however, either the primary or the secondary winding is mesh connected as shown in Fig. 5, triple harmonic currents can circulate round the mesh.

A further method for providing a path for the third harmonic current is to include a third or " tertiary " winding on the cores of the transformer, this winding being connected in mesh so that it forms a short-circuited path for the triple harmonic currents as shown in Fig. 6.

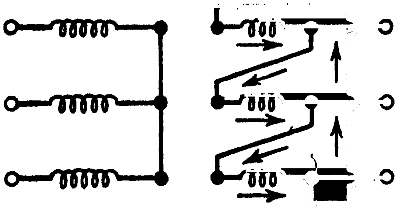


Fig. 5

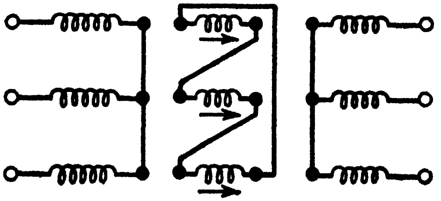


Fig. 6

6.—What will be the effect on the harmonics of the magnetising current wave of a transformer if a third harmonic appears in the flux wave ?

Answer.—In order to produce a sine wave of back e.m.f. when the applied p.d. wave is sinusoidal and the transformer core is worked at a high value of the flux density, it has been seen in Examples 2 and 3 that the magnetising current wave must contain harmonics, and these become very pronounced when the flux density reaches such very high values as about 16,000 gauss. If, however, one of the current harmonics of the magnetising current wave were absent, such, for example, as would be the case when the transformer windings are both connected in star (see Example 5), since there would be no return path for such a harmonic, then a harmonic will appear in the flux wave, the order of this flux harmonic being the same as the order of the missing current harmonic. That this must be so is seen from the following considerations. When the third harmonic of the current wave is flowing and the flux wave is sinusoidal, this implies that the current harmonic is suppressing a flux harmonic of the same order but of opposite sign, so that when this restraining current harmonic is absent, the previously suppressed flux harmonic is free to develop. That is to say, if the third harmonic of the current is negative, as in Fig. 2, Example 2, then when this current harmonic is absent, a positive third harmonic will appear in the flux wave.

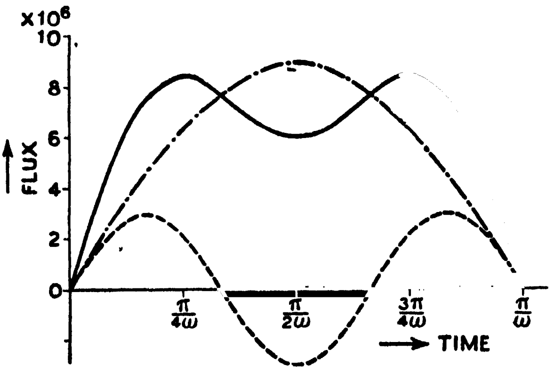


Fig. 7

In Fig. 7 is shown a flux wave which, in addition to the fundamental sine wave, has a large positive third harmonic, and the reason for choosing such a large third harmonic component will appear later. If the magnetising current wave corresponding to the flux wave of Fig. 7 be now obtained with the aid of the magnetisation curve of Fig. 1*b* in Example 2, as already explained, then the current wave of Fig. 8 will be the result. If this wave is analysed it will be found to contain in addition to the fundamental sine wave, third and fifth harmonics of the following magnitudes, viz.,

$$A_1 = 2{\cdot}56 : A_3 = 1{\cdot}4 ; A_5 = -0{\cdot}7.$$

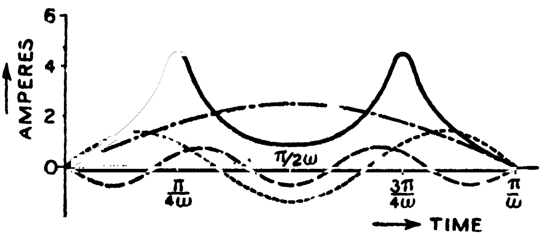


Fig. 8

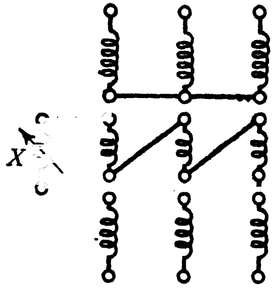


Fig. 9

It will be seen therefore that the introduction of a large third harmonic in the flux wave has caused the reversal of the sign of both the third and fifth harmonics of the current wave. For purposes of comparison the analysis of the current wave of Example 3 is re-stated here, that is, $A_1 = 5{\cdot}15 : A_3 = -3{\cdot}07 : A_5 = 1{\cdot}84$.

The general result is thus obtained that, when no harmonic appears in the flux wave, the magnetising current wave contains a *negative* third and a *positive* fifth harmonic, whilst if a large *positive* third harmonic appears in the flux wave, the corresponding current wave contains a *positive* third and a *negative* fifth harmonic. It follows, therefore, that by suitably choosing the magnitude of the third harmonic which is allowed to appear in the flux wave, either the third or the fifth harmonic of the current wave can be eliminated, and it is possible also to obtain the condition that when one of these harmonics is zero the other is very small. This very important result forms the basis for eliminating the fifth and seventh harmonics from the magnetising current wave, and the following two methods may be considered :—

(i) A tertiary winding on the transformer cores is connected in mesh, but instead of being short-circuited on itself it is closed through a choking coil *X*, of which the magnitude of the reactance can be adjusted (see Fig. 9). In this way, the magnitude of the third harmonic of current which is developed in the tertiary winding can be controlled so that the current harmonic which it is desired to suppress disappears from the wave form. This method is due to Buch and Heuter.

(ii) Instead of controlling the magnitude of the third harmonic of current, an alternative is to control the third harmonic of flux. In this case the transformer has five cores instead of three (see Fig. 10). The primary and secondary windings are arranged on the three central cores, whilst the two outer cores are left unwound. It will be seen that, just as in the case of the third harmonic of the current wave, which cannot flow in supply mains because there is no return path for it, so in the case of the third harmonic of flux in a three-core transformer this flux harmonic cannot develop because it would have no return (iron) path). When a fourth and fifth core are provided, however, these extra cores form a return path for the flux of the third harmonic. By suitably proportioning the cross section of these supplementary cores, the magnitude of the third harmonic of flux can be controlled and, consequently, the fifth and seventh harmonics eliminated from the current wave form of the magnetising current. Fig. 11 shows oscillographic records of the way in which this method operates to improve the wave form of the magnetising current, and these records refer to the conditions that the flux-density in the transformer core reaches the high value of 15,000 gauss. Fig. 11*a* shows the wave form of the magnetising current when the transformer had only the normal number of three cores. Fig. 11*b* shows the wave form of the magnetising current for the same transformer when a fourth and fifth core were added in accordance with the principle explained in the foregoing with reference to Fig. 10, the cross section of each of these additional cores being (in the first place) the same as that of the three normal cores. It is to be observed that in each of the oscillograms of Figs. 11*a* and 11*b* a prominent fifth harmonic is present,

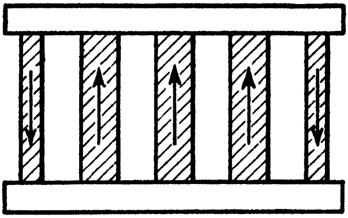
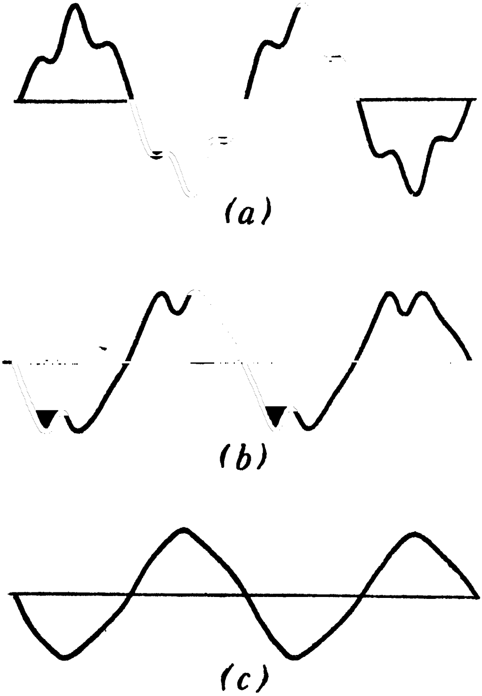


Fig. 10



(*a*)

(*b*)

(*c*)

Fig. 11

but that the fifth harmonic of Fig. 11*a* is 180° out of phase with the fifth harmonic of Fig. 11*b*. That is to say, by providing a return path by means of two additional cores of full cross section, the fifth-harmonic of the magnetising current wave form became reversed in sign. When the section of the two additional cores was suitably restricted, the oscillogram of the magnetising current shown in Fig. 11*c* was obtained, and this is seen to be of practically sinusoidal wave form. See also (T.F.W.) " Electrical Times," Dec. 21, 1939, p. 629 ; " Beama " Journal, Oct., 1940, p. 60.

7.—Draw a diagram showing the flux density distribution in the air-gap of a single-phase concentrated two-pole winding. The radial length of the air-gap is δ cm. and the air-gap surfaces of both stator and rotor cores are smooth cylindrical surfaces. Show in the diagram the fundamental sine wave and the third harmonic of flux density distribution in the air-gap.

ANSWER.—The single-phase concentrated winding is shown by the conductors *S* and *C* in Fig. 12. If the reluctance of the iron part of the magnetic circuit is neglected, the flux density in the air-gap at any moment will be given by the expression

$$B_t = \mu H_t = \frac{2\pi}{10} \frac{iw}{\delta} \text{ gauss}$$

since $\mu = 1$ for the air-gap. It is also to be observed that for any such magnetic path as is shown by the dotted circuit in Fig. 12, the length of path in the air-gap is 2δ cms.

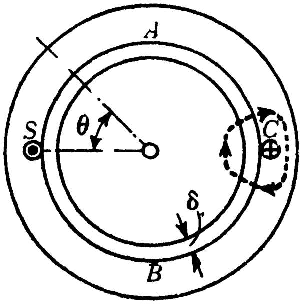


Fig. 12

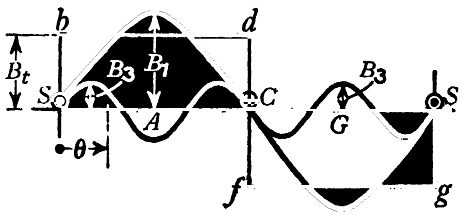


Fig. 13

In this expression *i* amperes is the value at any moment *t* of the alternating current in the winding *S C* and *w* the number of turns. This expression shows that the flux density at any moment *t* has the same value B_t everywhere in the air-gap, so that if it be assumed that the stator core is cut radially at, say, some place between *S* and *A* in Fig. 12 and then opened out, the surface of the gap will be seen as the horizontal straight line of Fig. 13. If values of the flux density in the gap are then plotted as ordinates, the rectangular diagram *SbdCfgS* will be obtained. In accordance with the result given on page 417 of "Principles of Electrical Engineering," expression 31, this rectangular wave can be analysed into a fundamental sine wave and a series of odd harmonics, of which only the fundamental sine wave and the third harmonic are shown in Fig. 13. The respective peak values of these component waves will be

$$B_1 = \frac{4}{\pi} B_t\,; \quad B_3 = \frac{1}{3}\,\frac{4}{\pi} B_t\,; \quad B_5 = \frac{1}{5}\,\frac{4}{\pi} B_t \text{ and so on.}$$

It will be seen that the fundamental wave of flux density will be the main two-pole field of the machine, whilst the third harmonic will produce a six-pole field, the fifth a ten-pole field, and so on. As the current in the stator winding alternates at a frequency of *f* cycles per second, all the component harmonic fields will alternate in magnitude with the same frequency, but their individual positions in the air-gap space will remain fixed.

8.—A three-phase two-pole concentrated stator winding is shown in Fig. 14, together with the time vector diagram of the three-phase alternating current which is supplied to the winding. Construct a diagram showing the three fundamental waves of magnetic flux density with the time as the abscissa axis, as well as the fifth harmonic waves associated with each of these fundamental waves. From this diagram derive the direction of rotation of the ten-pole magnetic field as referred to that of the two-pole field. Find the speed of rotation of the ten-pole field in terms of the speed of rotation of the two-pole field.

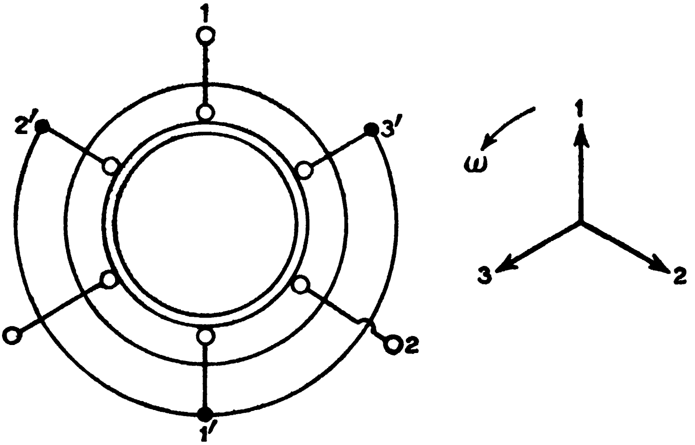


Fig 14

ANSWER.—The arrangement of the stator windings is shown in Fig. 14. The current in each phase winding will produce a rectangular flux density distribution which may be analysed into a fundamental wave and a series of harmonics as shown in the solution to Example 7 (see also page 417, Chapter XIII). Fig. 15*a* shows the three fundamental sine waves of alternating magnetic flux density distribution as a function of the time, but for purposes of clarity, only the positive half-waves are drawn in. It will be seen from this diagram that the waves reach their respective maximum positive values in the order I, II, III. In Fig. 15*a* is also shown the fifth harmonic of flux density distribution associated with the fundamental wave I, the magnitude of this fifth harmonic being shown

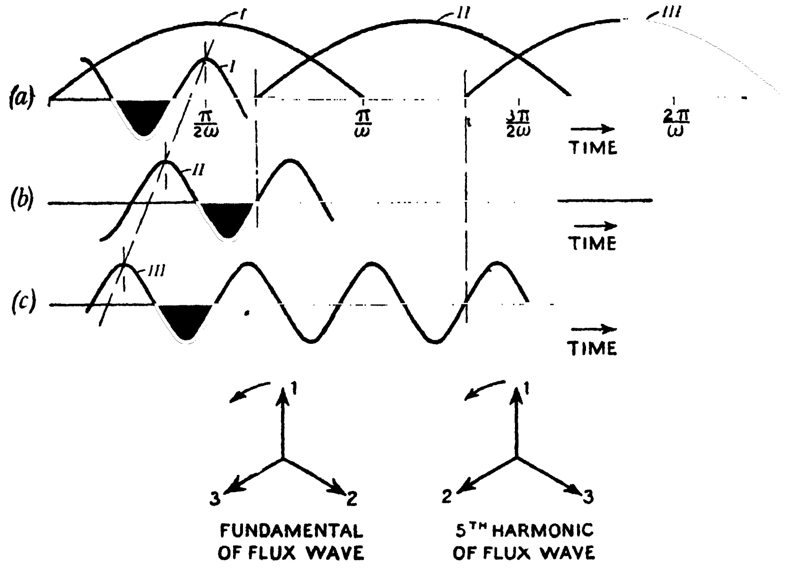


Fig 15

out of scale for purposes of clarity. In Figs. 15*b* and 15*c* are shown the fifth harmonic waves for the fundamental waves II and III, respectively, and it will be seen from these diagrams that the three individual sine waves of the fifth harmonic reach their respective positive maximum values in the order III, II, I. It follows therefore from these results that the ten-pole magnetic field due to the fifth harmonic rotates in the direction which is *opposite* to that of the two-pole rotating field due to the fundamental waves I, II, III.

If a similar set of diagrams be drawn out for the seventh harmonic of flux density distribution, it will be found that this 14-pole field will rotate in the *same* direction as the fundamental two-pole field. If corresponding diagrams were to be drawn out for the harmonic numbers 3, 9, 15, and any other odd multiple of 3, it would be found, for example, that all the six poles of the third harmonic field would coincide in position for all three phases of the stator winding so that this would give rise to a magnetic field which would alternate in magnitude with the frequency f of the fundamental wave but would not rotate.

It is further to be noted for example, with regard to the ten-pole field of the fifth harmonic, that since the sinusoidal fields due to each of the three phases alternate with the frequency f of the current in the stator winding, the rotating field only progresses a distance corresponding to one pair of poles for each cycle of the alternating current, so that the speed of rotation of the ten-pole field due to the fifth harmonic will be only one-fifth that of the two-pole field. Similarly, for the rotating magnetic field due to the nth harmonic (which is not a multiple of 3), the speed of rotation will be $\frac{1}{n}$th that of the fundamental two-pole rotating field.

9.—If a balanced three-phase star-connected, inductive system is connected in parallel with a balanced three-phase, mesh-connected, inductive system, show that, when the cores of the inductances are operating at saturation values of the flux densities, the consequent fifth and seventh harmonics in the individual phases of each system can be eliminated from the line currents.

Answer.—When a three-phase supply system is connected to a mesh-connected load of three balanced choking coils as shown in Fig. 16, then if the currents are of such a magnitude that the iron cores become magnetised to saturation, the current wave form will be highly distorted as shown for instance in Fig. 2 of Example 2, so that relatively large current harmonics will flow in each coil. The line current for such a mesh-connected load is related to the current in the mesh in accordance with the vector diagram of Fig. 16, Chapter XI, page 357, in which the vector of the line current I_A has been obtained by taking the vector difference of the phase currents I_{CA} and I_{AB}, viz.,

$$-(I_A)_1 = (I_{CA})_1 - (I_{AB})_1$$

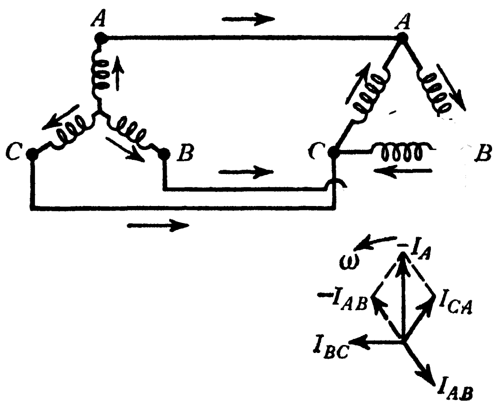


Fig. 16

The same result is shown in Fig. 17*a* in a somewhat different way. In this diagram, the fundamental waves of the phase currents $(I_{CA})_1$ and $-(I_{AB})_1$ are shown as functions of the time by means of broken line sine waves, and the corresponding line current is obtained by adding corresponding ordinates of the two sine waves, this being shown by the full-line sine wave (Fig. 17*a*) of the fundamental frequency.

Since the flux densities in the individual cores of the inductances will reach saturation values, pronounced current harmonics will appear in each phase of the mesh-connected inductances. The fifth harmonic current for the phase *C A* and the reversed phase current of *A B*, viz.,

$$(I_{CA})_5 \text{ and } -(I_{AB})_5$$

are shown by the two full-line curves in Fig. 17*c*, the resultant chain-dotted sine wave being so obtained.

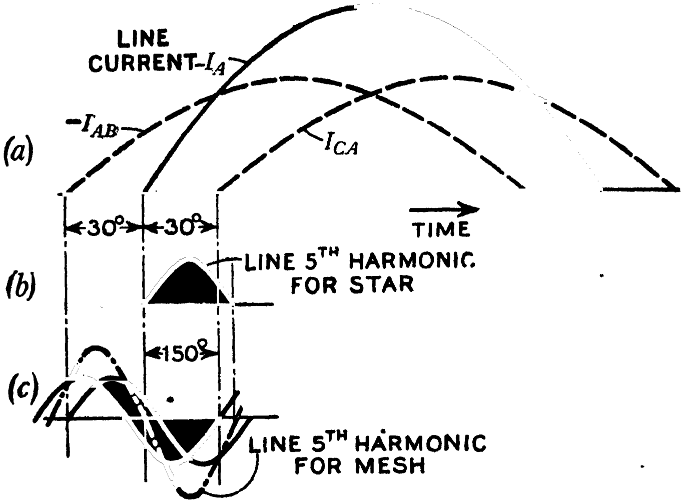


Fig. 17

Now suppose that, to the same supply system, a balanced three-phase star-connected system of choking coils is connected, as shown in Fig. 18, that is to say the mesh-connected system is connected in parallel with a star-connected system (as shown in Fig. 19). It will be clear that the current in phase *A* of the star-connected system will be in phase with the line current which supplies that phase, since, of course, the line current I_A is also the phase current of the star to which that line is connected. This line current for the fifth harmonic is shown by the full-line sine wave of Fig. 17*b*, and it is at once seen by comparing Figs 17*b* and 17*c* that the line current of the fifth harmonic supplying the mesh is in direct opposition to the line

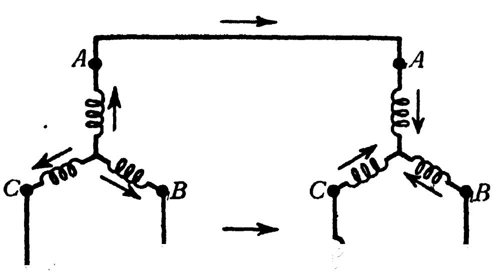


Fig. 18

current of the fifth harmonic which supplies the star system. It follows, therefore, that by suitably proportioning the mesh and star systems, the fifth-harmonic currents can be made equal, and since they are 180° out of phase with each other in the line, the result is obtained that the line is relieved of the fifth-harmonic current which is required to produce the current wave form of Fig. 2 (Example 2 of this Test Paper), and this harmonic current will simply circulate between the star and mesh windings and will not appear in the line at all (Fig. 19). The corresponding vector diagram for the fifth harmonic of the parallel star-mesh system is shown in Fig. 20. Similar considerations lead to the same result as regards the elimination of the seventh harmonic.

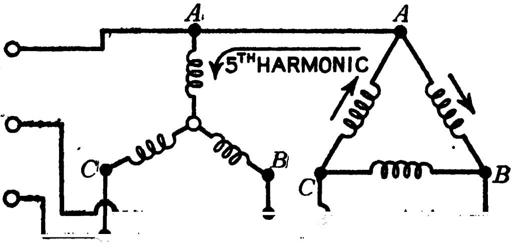


Fig. 19

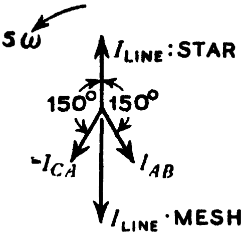


Fig. 20

This important principle has been applied in recent years to the design of large transformers to enable them to operate at high value of the core flux densities—up to 16,000 gauss—and still to maintain a practically pure sine wave form of current in the supply lines. (See also " Electrical Times " and " Beama " Journal : loc. cit.)

10.—Describe some method by means of which the magnitudes of the individual harmonics in a given wave form of e.m.f. may be measured.

ANSWER.—The operation of the " harmonic analyser," which is described in what follows, is based on the principle of action of the ordinary dynamometer type of wattmeter, it being well known that such an instrument can only respond when the currents in the fixed and moving coils are of the same frequency.

The pressure wave form which it is desired to analyse is connected across the terminals A and B Fig. 21, and in series with the moving coil b of the dynamometer is connected a condenser C_1. If the applied pressure wave E contains an n^{th} harmonic of r.m.s. value E_n volts, then the current I_b which will flow in the moving coil will also contain an n^{th} harmonic of magnitude

$$I_n = \frac{E_n}{Z_n}$$

where, $$Z_n = \sqrt{r_1^2 + \left\{2\pi nfL_1 - \frac{1}{2\pi nfC_1}\right\}^2}$$

and f is the frequency of the fundamental wave of the applied p.d.

The analysing current I_a is obtained by means of a synchronous motor, on the shaft of which is a contact disc *S*. On the front surface of this disc concentric rings are arranged, each ring being built up of metal segments separated by alternating with insulating segments, one such ring of segments being provided for each harmonic which it is desired to measure. The adjustable brush *K* is set so as to be in contact with the ring which corresponds to the n^{th} harmonic, and the grid of the thermionic valve then receives *n* impulses of pressure per second. The current which will flow in the anode circuit will consist of a direct current upon which is superposed a heavily distorted wave of fundamental frequency *n*.f.

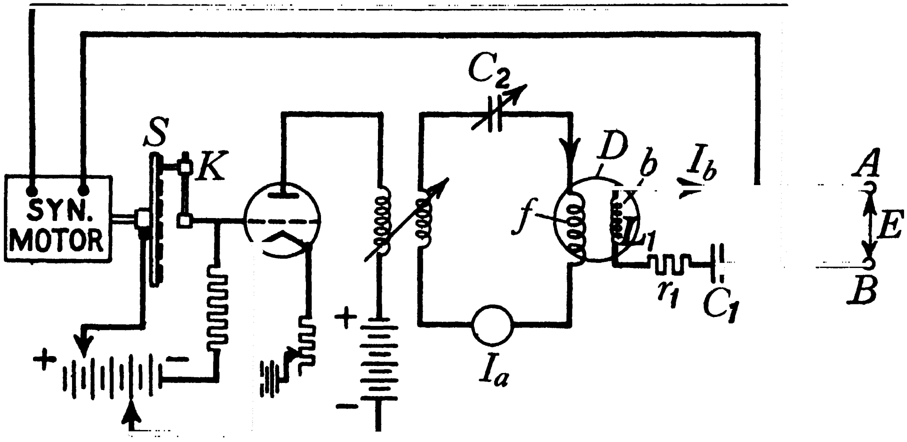


Fig. 21

By means of an oscillatory circuit, which is tuned to the frequency of the n^{th} harmonic, an almost pure sine wave of current I_a is obtained, the frequency of which is *n.f*. This current flows in the fixed coil of the dynamometer and the deflection of the instrument will then be

$$\alpha = kI_aI_n \cos \varphi$$

where φ is the phase displacement between the currents I_a and I_n, and by suitably adjusting the position of the brush *K* on the contact disc, this phase displacement can be reduced to zero. The instrument will then read

$$\alpha_{max} = kI_aI_n$$

The analysing current *Ia* is measured by means of a hot-wire ammeter, and if the constant *k* of the dynamometer is known, the magnitude of the *nth* harmonic current is

$$I_n = \frac{\alpha_{max}}{kI_a}$$

and consequently the *nth* harmonic of the pressure wave is

$$E_n = I_n Z_n$$

It is stated that the magnitude of a 35th pressure harmonic has been measured in this way, the value of which is only $\frac{1}{10{,}000}$th part of the fundamental pressure wave. (See Walker, Tyac and Coe: Journal I.E.E., Vol. 63; 69.)

11.—What is meant by the "frequency spectrum" of a non-sinusoidal wave form?

ANSWER.—The frequency spectrum is a graphical representation of the harmonic analysis of a non-sinusoidal wave form. The abscissa represents the order *n* of the individual sinusoidal harmonic waves and the ordinates represent the magnitude of these harmonics. For example, the harmonic analysis of the rectangular wave form of Fig. 22*a* has been shown to be (page 417):

$$\frac{4}{\pi} H \left[\sin x + \tfrac{1}{3} \sin 3x + \tfrac{1}{5} \sin 5x + \dots\dots \right]$$

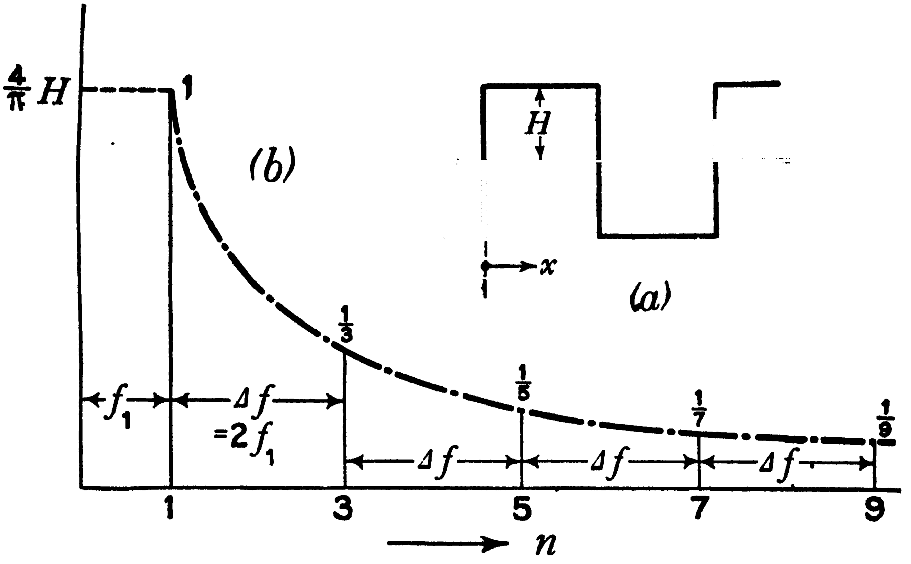


Fig. 22

In Fig. 22*b* the order of the individual harmonics are marked 1, 3, 5, 7 . . . and the amplitudes of the individual sine waves are shown by the ordinate scale. Thus, for example, the amplitude of the *n*th harmonic is $\frac{1}{n}$th that of the fundamental wave.

The difference of frequency between two successive harmonics is:

$$\triangle f = 2 f_1$$

where f_1 is the frequency of the fundamental wave.

The broken-line curve of Fig. 22*b* which joins the extremities of the ordinates for the individual harmonics is seen to be a rectangular hyperbola.

Chapter XIV

THE PENETRATION OF ALTERNATING MAGNETIC FLUX AND ALTERNATING CURRENT (SKIN EFFECT)

1.—*Find the diameter of the iron rod which will satisfy the following conditions, viz.:*

$$\frac{\textit{Resistance to a.c.}}{\textit{Resistance to d.c.}} = 6: \ \textit{frequency}\ f = 2{,}000\ \textit{hz.}:$$

$$\rho = 12 \times 10^{-6}\ \textit{ohm per cm./cm}^2.;\ \textit{permeability}\ \mu = 500.$$

ANSWER.—

$$Y = 2\sqrt{2}\,\pi\sqrt{\frac{\mu f}{\rho \times 10^9}} = 2\sqrt{2}\,\pi\sqrt{\frac{500 \times 2{,}000}{12 \times 1{,}000}} = 81$$

From expression 49, page 452 " Principles of Electrical Engineering," the equivalent depth of penetration is

$$\sigma = \frac{1}{2\pi}\sqrt{\frac{\rho.10^9}{\mu.f}}$$

that is,

$$\sigma = \frac{\sqrt{2}}{Y} \doteq 0{\cdot}0175 \text{ cm.}$$

Hence $\frac{a}{2\,\sigma} = 6$ where a cm. is the radius

$$\therefore \text{ the diameter is } d = 2a = (6 \times 2\,\sigma)\,2$$

that is $$d = 24 \times 0{\cdot}0175 = \underline{0{\cdot}42 \text{ cm.}}$$

2.—*At what frequency will a straight copper wire, 1 mm. diameter, have a resistance 4 times its d.c. value ?*

ANSWER.—Since the given conditions imply a large value of Ya the relationship for the depth of penetration σ as given on page 454 of " Principles of Electrical Engineering" may be used, viz.:

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = \frac{a}{2\sigma} = 4;$$

so that $$\sigma = \frac{a}{8} = \frac{0\cdot05}{8} = \underline{0\cdot00625 \text{ cm.}}$$

that is $$Y = \frac{\sqrt{2}}{\sigma} = \frac{\sqrt{2}}{0\cdot00625} = 226$$

and hence $$Y = 2\sqrt{2}\,\pi\sqrt{\frac{f}{1\cdot77 \times 10^3}} = 226$$

from which $$\underline{f = 1\cdot14 \times 10^6 \text{ hz.}}$$

3.—***An iron wire carries a.c. at a frequency of 5,000 hz. Find what the diameter will be if the resistance to a.c. is 4 times the d.c. value. The specific resistance is $\rho = 12 \times 10^{-6}$ ohm/cm./cm.²***

Answer.— $$Y = 2\sqrt{2}\,\pi\sqrt{\frac{2{,}000 \times 500}{12 \times 10^3}} = \frac{2\sqrt{2}\,\pi}{\sqrt{12}}\sqrt{1{,}000} = 80\cdot8$$

and $$\sigma = \frac{\sqrt{2}}{Y} = 0\cdot01753 \text{ cm.}$$

so that $$\frac{a}{2\sigma} = 4 : a = 8\sigma = 0\cdot140 \text{ cm.}$$

and the diameter is $2a = 0\cdot28$ cm.

4.—***Find the current distribution over the section of a straight copper wire, 1 mm. diameter, for a frequncy of 0.25 × 10⁶ hz., if $\rho = 1.77 \times 10^{-6}$ ohm/cm./cm.² Draw the vector diagram for the current distribution.***

Answer.—$Y = 0\cdot211\sqrt{f} = 105\cdot5 : Ya = 105.5 \times 0\cdot05 = 5\cdot275$

$$\dot{i}_o = \dot{i}_{\max}\frac{1}{J_o\{\sqrt{-j}\,5\cdot275\}} = \frac{\dot{i}_{\max}}{7\cdot25\,e^{j188^\circ}}$$

Divide the section of the wire into 10 concentric bands, each of radial thickness 0·05 mm. = 0·005 cm., as shown in Fig. 1. Then for
Band 1 : mean radius $x_1 = 0\cdot05 - 0\cdot0025 = 0\cdot0475$ cm.

$$Yx_1 = 105\cdot5 \times 0\cdot0475 = 5$$

$$\dot{i}_{x_1} = \dot{i}_{\max}\frac{J_o\{\sqrt{-j}\,5\}}{J_o\{\sqrt{-j}\,Ya\}} = \dot{i}_{\max}\left[\frac{6\cdot2\,e^{j180^\circ}}{7\cdot25\,e^{j188^\circ}}\right]$$

$$\underline{= 0\cdot855\,\dot{i}_{\max}\,e^{-j8^\circ} \text{ amperes per sq. cm.}}$$

Area of Band 1 $= 2\pi \times 0{\cdot}0475 \times 0{\cdot}005 = 0{\cdot}00149$ cm., and the current in Band 1 is therefore $= 0{\cdot}00149 \times 0{\cdot}855 i_{\max}$

$$= \underline{0{\cdot}001372\, i_{\max}\, e^{-j8^\circ}} \text{ amperes.}$$

and this gives the current vector $O1$ of Fig. 2.

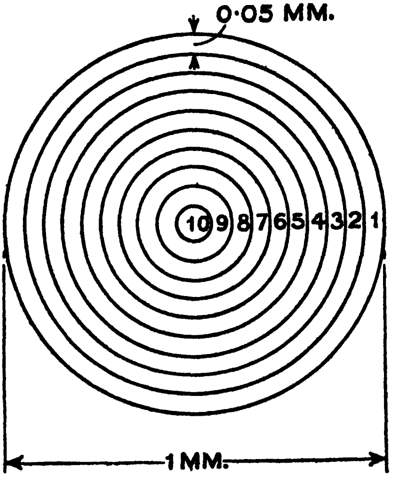


Fig. 1

Proceeding in this way for each of the 10 bands of Fig. 1, the following Table has been prepared :—

Band Number	$Y{\cdot}x$	Vector of Current Density $\dot{i}_x$	Sectional Area cm²	Vector of Current in Band
1	5	$0{\cdot}855 i_{\max}\, e^{-j8^\circ}$	0·00149	$0{\cdot}00137 i_{\max}\, e^{-j8^\circ}$
2	4·46	$0{\cdot}621 i_{\max}\, e^{-j29.3}$	0·00133	$0{\cdot}000832 i_{\max}\, e^{-j29.3^\circ}$
3	3·93	$0{\cdot}449 i_{\max}\, e^{-j58^\circ}$	0·00128	$0{\cdot}000574 i_{\max}\, e^{-j58^\circ}$
4	3·42	$0{\cdot}338 i_{\max}\, e^{-j74.6^\circ}$	0·00102	$0{\cdot}000346 i_{\max}\, e^{-j74.6^\circ}$
5	2·90	$0{\cdot}254 i_{\max}\, e^{-j95.8^\circ}$	0·000864	$0{\cdot}000219 i_{\max}\, e^{-j95.8^\circ}$
6	2·36	$0{\cdot}195 i_{\max}\, e^{-j120^\circ}$	0·000706	$0{\cdot}000138 i_{\max}\, e^{-j120^\circ}$
7	1·84	$0{\cdot}162 i_{\max}\, e^{-j142.5^\circ}$	0·00055	$0{\cdot}000089 i_{\max}\, e^{-j142.5^\circ}$
8	1·31	$0{\cdot}143 i_{\max}\, e^{-j164.3^\circ}$	0·00039	$0{\cdot}000056 i_{\max}\, e^{-j164.3^\circ}$
9	0·79	$0{\cdot}139 i_{\max}\, e^{-j179^\circ}$	0·000236	$0{\cdot}000033 i_{\max}\, e^{-j179^\circ}$
10	0·26	$0{\cdot}138 i_{\max}\, e^{-j187^\circ}$	0·00008	$0{\cdot}000011 i_{\max}\, e^{-j187^\circ}$

The vector diagram showing the vectors of total current in the 10 bands as given by the last column of this Table is drawn in Fig. 2. The corresponding vector polygon is given in Fig. 3, from which the vector of the total current in the wire is obtained, viz. :—

$$\underline{OA = 0{\cdot}00288 i_{\max}\, e^{-j38\frac{1}{4}°}}$$

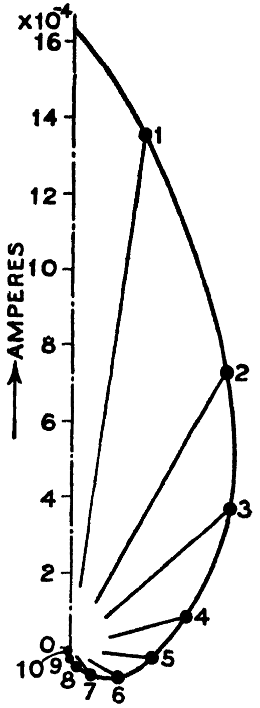


Fig. 2

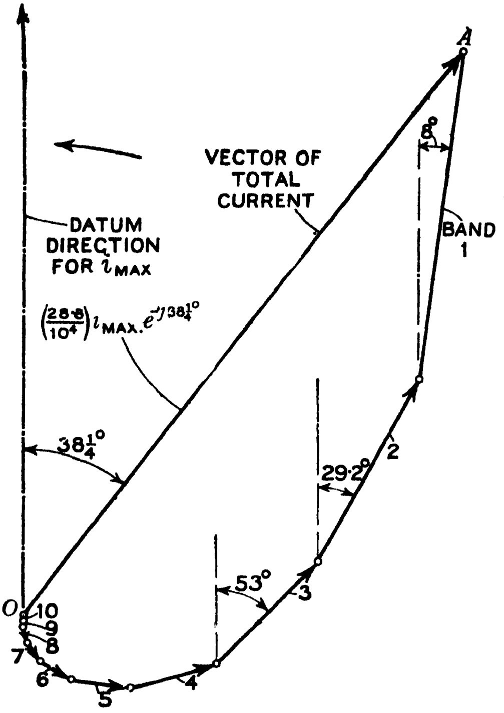


Fig. 3

The calculated value of the total current vector is

$$\mathfrak{I}_T = \frac{2\pi \times 0{\cdot}05}{105{\cdot}5}\, i_{\max}\, \frac{J_1\{\sqrt{-j}\,Ya\}}{J_0\{\sqrt{-j}\,Ya\}}\, e^{j45°}$$

$$= 0{\cdot}002995 i_{\max}\, \frac{6{\cdot}8 e^{j103\frac{1}{4}°}}{7{\cdot}25 e^{j188°}}\, e^{j45°}$$

that is $$\underline{\mathfrak{I}_T = 0{\cdot}00281(i_{\max}\, e^{-j39\frac{1}{2}°})}$$

The current taken by the wire from the supply therefore lags by 39½° on the applied p.d., i.e., on the vector of current density $i_{\max}$ in the surface skin. Since the skin layer does not link with any part of the magnetic flux which is developed in the section of the conductor, the current density in the surface skin must be in phase with the applied p.d. It is to be noted (as pointed out on page 434 of "Principles") that the magnetic flux which is developed outside the wire produces a purely reactive back e.m.f. and has no effect on the power supplied to overcome the watt loss in the resistance. This reactive back e.m.f. has not been included in this investigation since it has no effect on the resistance of the conductor to a.c.

If V r.m.s. volts is the applied pressure, the current density i_{max} in the surface skin will be in phase with V so that if the applied pressure were d.c., the consequent d.c. current would be $i_{\text{max}}\ \pi a^2$ where πa^2 is the cross sectional area of the conductor. For a.c. the effective impedance Z is related to the magnitude of the total current by the equation

$$V = \dot{I}_{\text{T}}Z$$

and since

$$V = (i_{\text{max}}\pi a^2)R_{\text{d.c.}}$$

it follows that

$$\frac{Z}{R_{\text{d.c.}}} = \frac{i_{\text{max}}\ \pi a^2}{I_{\text{I}}}$$

But the effective resistance to a.c. is $Z \cos \varphi$ where $\varphi = 38\frac{1}{4}°$ in the present example. Hence

$$\frac{R_{\text{a.c.}}}{R_{\text{d.c.}}} = \frac{Z\cos\varphi}{R_{\text{d.c.}}} = \frac{i_{\text{max}}\pi a^2 \cos\varphi}{I_{\text{I}}}$$

and, since $\pi a^2 = \pi \times (0{\cdot}05)^2 : \varphi = 38\frac{1}{4}° : I_{\text{I}} = 0{\cdot}00283 i_{\text{max}}$

$$\frac{R_{\text{a.c.}}}{R_{\text{d.c.}}} = \frac{\pi \times 0{\cdot}0025 \cos 38\frac{1}{4}°}{0{\cdot}00283} = 2{\cdot}18$$

The exact theoretical calculation of the value of the ratio $\frac{R_{\text{a.c.}}}{R_{\text{d.c.}}}$ for the conditions of this example is as follows:

$$\frac{R_{\text{a.c.}}}{R_{\text{d.c.}}} = \left[\left\{\frac{Ya}{2}\frac{J_0\{\sqrt{-j}\,|Ya\}}{J_1\{\sqrt{-j}\,|Ya\}}\right\}e^{-j\,45°}\right]\cos\varphi$$

$$= \left[2{\cdot}63\,\frac{7{\cdot}2\,e^{j\,188°}}{6{\cdot}78e^{j\,103\frac{1}{2}°}}\,e^{-j\,45°}\right]\cos\varphi$$

that is

$$\frac{R_{\text{a.c.}}}{R_{\text{d.c.}}} = \left[2{\cdot}79\,e^{j\,39\frac{1}{2}°}\right]\cos 39\frac{1}{2}° = \underline{2{\cdot}14}$$

5.—Alternating current at 5,000 frequency is passed through an iron lamination to which the following numerical data apply:

thickness $2\triangle = 0.5$ mm.: $\mu = 2{,}000$: $\rho = 10 \times 10^{-6}$ ohm/cm./cm².

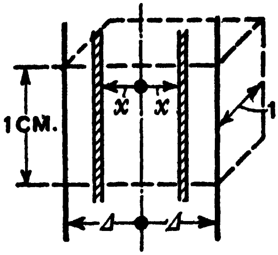


Fig. 4

Plot a curve of current distribution for the following moments:

(i) The current in the surface skin has its maximum value.

(ii) The current in the surface skin is zero.

ANSWER.—Fig. 4 shows a sketch of a section of the iron lamination. At the distance x cm. from the median plane the current density is:

$$i_x = i_{\max} \frac{\operatorname{Cosh} \mathfrak{S} x}{\operatorname{Cosh} \mathfrak{S} \triangle} \text{ amperes per sq. cm.}$$

Where $\mathfrak{S} = \alpha + j\,\alpha$: and $\alpha = \sqrt{\dfrac{2\pi\,\mu\omega}{\rho}}$: that is,

$$\alpha = \sqrt{\frac{2\pi \times 2{,}000 \times 31{,}400}{10^4}} = 198{\cdot}5 : \alpha\triangle = 4{\cdot}97 \text{ radians} = 284^\circ$$

The length of the wave of current penetration is

$$\lambda = \frac{2\pi}{\alpha} = \frac{2\pi}{198{\cdot}5} = 0{\cdot}0316 \text{ cm.}$$

Velocity of travel of the wave is:

$$c = \lambda f = 0{\cdot}0316 \times 5{,}000 = 158 \text{ cms. p. sec.}$$

$$\begin{aligned}\operatorname{Cosh} \mathfrak{S}\triangle &= \operatorname{Cosh}(\alpha\triangle + j\,\alpha\triangle) = \operatorname{Cosh}(4{\cdot}97 + j\,4{\cdot}97)\\ &= \operatorname{Cosh} 4{\cdot}97 \cos 4{\cdot}97 + j \operatorname{Sinh} 4{\cdot}97 \sin 4{\cdot}97\\ &= \underline{71{\cdot}2\, e^{-j\,76.2^\circ}}\end{aligned}$$

$$i_x = \frac{i_{\max}}{71{\cdot}2\, e^{-j\,76.2^\circ}}\,\frac{1}{2}\left\{e^{ax}\;e^{jax} + e^{-ax}\;e^{-jax}\right\}$$

neglecting the reflected wave as insignificantly small, that is, neglecting the expression $\frac{1}{2}\,(e^{-ax}\;e^{-jax})$, the forward wave of current density penetration is—

$$i_x = \frac{i_{\max}}{142{\cdot}4\, e^{-j\,76{\cdot}2^\circ}}\left\{e^{ax}\;e^{jax}\right\} = \underline{\frac{i_{\max}\, e^{j\omega t}}{142{\cdot}4\, e^{-j\,76{\cdot}2^\circ}}\left\{e^{ax}\;e^{jax}\right\}}$$

(i) $t = o$: current in the surface skin is a maximum.

$$\underline{x = \triangle} : \dot{\mathrm{i}}_\Delta = \frac{\dot{i}_{\max}}{142{\cdot}4\, e^{-j\, 76.2^\circ}} \left\{ e^{a\Delta} \;\; e^{ja\Delta} \right\} = \frac{\dot{i}_{\max}}{142{\cdot}4\, e^{-j\, 76.2^\circ}}\, e^{4.97}\, e^{j\, 284^\circ}$$

$$\text{i.e., } \underline{\dot{\mathrm{i}}_\Delta = \dot{i}_{\max}.}$$

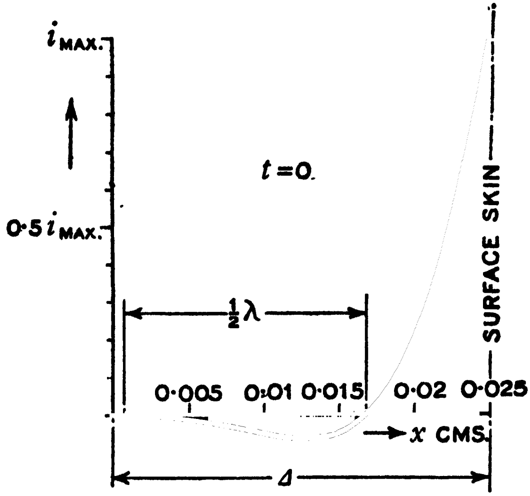


Fig. 5

$$\underline{x = 0{\cdot}9\,\triangle} : \dot{\mathrm{i}}_x = \frac{\dot{i}_{\max}}{142{\cdot}4\, e^{-j\, 76.2^\circ}} \left\{ e^{4.47}\, e^{j\, 256^\circ} \right\} = \frac{\dot{i}_{\max}\, e^{j\, 333.2^\circ}}{142{\cdot}4 \times 0{\cdot}0116}$$

$$\mathrm{i}_x = 0{\cdot}605\, \dot{i}_{\max} \cos 27{\cdot}8^\circ = \underline{0{\cdot}53\ \dot{i}_{\max}}$$

and so on, thus giving the current density distribution curve of Fig. 5

(*ii*) $t = \dfrac{\pi}{2\omega}$: *current in the surface skin is zero.*

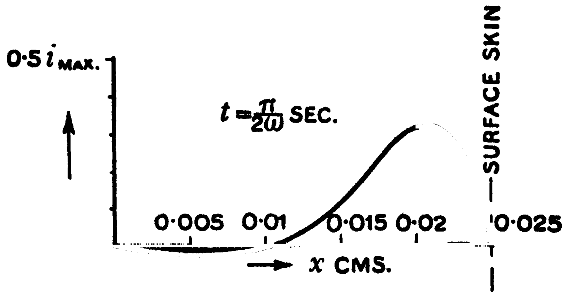


Fig. 6

$$x = \triangle : \dot{\mathrm{i}}_\Delta = \frac{\dot{i}_{\max}\, e^{j\frac{\pi}{2}}\, e^{a\Delta}\, e^{j\, a\Delta}}{142{\cdot}4\, e^{-j\, 76.2^\circ}} = \frac{\dot{i}_{\max}\, e^{4.97}\, e^{j\,(284+90)^\circ}}{142{\cdot}4\, e^{-j\, 76.2^\circ}} = \underline{0}$$

$$\underline{x = 0{\cdot}9\,\triangle} : i_x = \frac{i_{\max}\, e^{4.47}\, e^{j\,(0.9 \times 284^\circ + 90^\circ)}}{142{\cdot}4\; e^{-j\,76.2^\circ}} = 0{\cdot}605\; i_{\max} \cos 61{\cdot}8^\circ$$

that is,

$$i_x = \underline{0{\cdot}29\; i_{\max}}$$

and so on, thus giving the current density distribution curve shown in Fig. 6.

6.—Find the " penetration depth " of alternating magnetic flux in laminated conductors as defined by the following data :

	$\rho\Omega$/cm./cm.^{2}	μ
Dynamo stampings . . .	***$1^8\ 10^{-5}$***	***2,000***
Permalloy	***0.2×10^{-4}***	***10,000***
Cast steel	***0.2×10^{-4}***	***1,000***

ANSWER.—The induction density at a distance x cms. from the median plane (Fig. 7) is

$$\mathfrak{B}_x = B_{\max} \frac{\text{Cosh}\, \mathfrak{S}\, x}{\text{Cosh}\, \mathfrak{S}\, \triangle}$$

where $B_{\max}$ is the induction density at the surface skin. This expression

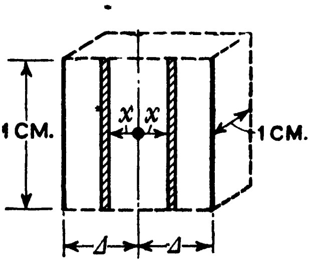


Fig. 7

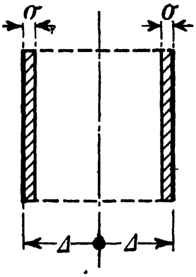


Fig. 8

is seen to be identical in form with that of expression 44, page 449 of " Principles," for the distribution of a.c. in laminated conductors so that the results already obtained can be applied immediately to the present problem. Consequently, the total magnetic flux in the lamination will be given by the expression :

$$2\int_0^{\triangle} \mathfrak{B}_x \,\alpha x = \frac{2\,B_{\max}}{\sqrt{2}\,\alpha} \sqrt{\frac{\text{Cosh}\; 2\,\alpha\,\triangle - \cos 2\,\alpha\,\triangle}{\text{Cosh}\; 2\,\alpha\,\triangle + \cos 2\,\alpha\,\triangle}} \;\ldots\ldots\ldots(\text{i})$$

When the product $\alpha\triangle$ is greater than about 4, the term cos 2 α becomes insignificant in comparison with Cosh $2\alpha\triangle$, so that the total magnetic flux of expression (i) for the two halves of the lamination then becomes

$$2\frac{B_{\max}}{\sqrt{2}\,\alpha}$$

That is to say, if a surface layer of thickness $\sigma = \frac{1}{\sqrt{2}\,\alpha}$ be considered for each half of the lamination, as shown in Fig. 8, and if it be assumed that the flux density is constant over the whole section of each of these strips and of magnitude $B_{\max}$, then the flux contained in these two strips will be equal to the total distributed flux in the lamination. The quantity σ is the equivalent " depth of penetration." That is,

$$\sigma = \frac{1}{\sqrt{2}\,\alpha} = \frac{1}{\sqrt{2}}\sqrt{\frac{\rho \times 10^9}{4\pi^2 \mu f}} = \sqrt{\frac{\rho \times 10^9}{8\pi^2 \mu f}}$$

If the corresponding numerical values of the given problem be inserted in this expression, then the equivalent depth of penetration will be found to be

$$\sigma = 0{\cdot}25\sqrt{\frac{1}{f}}\text{ cm. for dynamo stampings}$$

$$\sigma = 0{\cdot}16\sqrt{\frac{1}{f}}\text{ cm. for Permalloy}$$

$$\sigma = 0{\cdot}50\sqrt{\frac{1}{f}}\text{ cm. for cast steel}$$

From these data the following Table has been derived giving the depth of penetration σ in cms. :

f hertz	25	50	500	5,000	500,000
Dynamo Sheet	0·05	0·033	0·011	0·0035	0·00035
Permalloy ..	0·032	0·023	0·008	0·0023	0·00023
Cast Steel ..	0·10	0·071	0·022	0·0071	0·00071

It is to be observed that the depth of penetration of magnetic flux for a laminated conductor may be expressed in the form

$$\sigma = \frac{1}{\sqrt{2}\,\alpha} = 3{,}570\sqrt{\frac{\rho}{\mu f}}$$

whereas, for the depth of penetration of alternating current the expression 49 obtained on page 452 is

$$\sigma = 5{,}030\sqrt{\frac{\rho}{\mu f}}.$$

The difference in the numerical factor of these two expressions is accounted for by the fact that the depth of penetration σ for alternating current is calculated on the basis of the watt component of the total current, whereas, in the case of the magnetic flux penetration σ is calculated of the bases of the total flux in the lamination. For the conditions for which the calculation of depth of penetration are valid, the watt component of the current is $\frac{1}{\sqrt{2}}$ times the total current (see page 443 and expression 31 of " Principles of Electrical Engineering ").

Suppose, for example, the depth of penetration of a.c. is y and that the consequent surface density is, $i_q = \frac{i_{\max}}{p}$ and is such that, when constant throughout the depth y would give a total current equal to the actual current, viz.,

$$i_{\max}\, 3{,}570\sqrt{\frac{\rho}{\mu f}}.$$

Then,

$$\left(\frac{i_{\max}}{p}.y\right) = i_{\max}\left(3{,}570\sqrt{\frac{\rho}{\mu f}}\right) = \text{total current supplied to the lamination,}$$

so that

$$\frac{y}{p} = 3{,}570\sqrt{\frac{\rho}{\mu f}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

But the depth of penetration y must also be such that the resistance loss is equal to the actual resistance loss, and this requirement gives a second equation for p and y, viz.,

$$\frac{\rho\left(\frac{i_{\max}}{p}y\right)^2}{y} = \rho\left(\frac{i_{\max}}{p}\right)^2 y = \text{known resistance loss.}$$

Since, as stated above, the watt component of the current is $\frac{1}{\sqrt{2}}$ times the total current, it follows that,

$$\left(\frac{i_{\max}}{p}\right)^2 y = V\frac{i_{\max}}{p}\frac{y}{\sqrt{2}}$$

and, substituting for $V = \rho\, i_{\max}$ gives :

$$\rho \left(\frac{i_{\max}}{p}\right)^2 y = \rho \frac{i^2_{\max}}{p} \frac{y}{\sqrt{2}}$$

that is, $$\frac{y}{p} = \frac{y}{\sqrt{2}}$$

or, $$p = \sqrt{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

and hence $$y = p \times 3{,}570 \sqrt{\frac{\rho}{\mu f}} = \sqrt{2} \times 3{,}570 \sqrt{\frac{\rho}{\mu f}}$$

that is, $$\underline{y = 5{,}030 \sqrt{\frac{\rho}{\mu f}}}$$

Hence, the equivalent surface layer—equivalent both as regards total current content and resistance loss—has a breadth (Fig. 9)

$$y = 5{,}030 \boxed{\sqrt{\frac{\rho}{\mu f}}}\ cm$$

and the current density throughout this equivalent depth is $\frac{1}{\sqrt{2}} i_{\max}$ where $i_{\max}$ is the density at the surface skin for the actually existing current distribution.

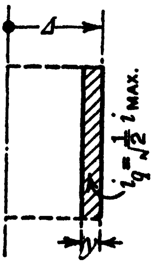


Fig. 9

7.—*Derive an expression for σ, the depth of penetration in a straight wire which is carrying high-frequency alternating current.*

ANSWER.—By definition the equivalent depth of penetration σ is such that if the current density is assumed to be constant throughout this depth and equal to $i_{\max.}$, i.e., the density at the surface when high frequency current is flowing, then (see Fig. 10) :

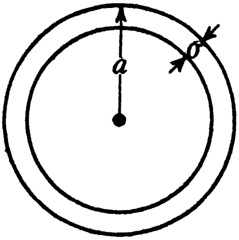


Fig. 10

$$\left(\frac{\text{Resistance of the surface skin}}{\text{Resistance of the whole section of the wire}}\right)$$

$$= \left(\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}}\right)$$

The resistance per cm. length of the skin layer $= \frac{\rho}{2\pi\, a\, \sigma}$ and the resistance per cm. length of the whole section of the wire is $\frac{\rho}{\pi\, a^2}$.

$$\text{hence, } \left(\frac{\text{Resistance of the surface skin)}}{\text{Resistance of the whole section of the wire}}\right) = \frac{a}{2\sigma} \ldots\ldots\ldots\ldots \text{(i)}$$

$$\text{Also } \frac{\text{Resistance to high frequency a.c.}}{\text{Resistance to d.c.}} =$$

$$\frac{Ya}{2}\left\{\text{The real component of } \frac{J_0\sqrt{-j}\,Ya}{J_1\sqrt{-j}\,Ya}\,e^{-j45^\circ}\right\}$$

For large values of Ya the real component in brackets becomes

$\cos 45^\circ = \dfrac{1}{\sqrt{2}}$ so that, for large values of Ya:

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = \frac{Ya}{2\sqrt{2}} = 0{\cdot}354\,Ya \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

The depth of penetration, therefore, is obtained by equating (i) and (ii), that is:

$$\frac{a}{2\sigma} = \frac{Ya}{2\sqrt{2}} \text{ or } Y\sigma = \sqrt{2}$$

$$\text{and, since } Y = \sqrt{\frac{8\pi^2\,\mu f}{\rho \times 10^9}} = \frac{2\,\pi\,\sqrt{2}}{3{\cdot}16 \times 10^4}\sqrt{\frac{\mu f}{\rho}}$$

$$\text{it follows that } \sigma = \frac{\sqrt{2}}{Y} = \frac{3{\cdot}16 \times 10^4}{2\sqrt{2}\,\pi}\sqrt{\frac{\rho}{\mu f}}$$

$$\text{that is, } \sigma = 5{,}030\sqrt{\frac{\rho}{\mu f}}$$

(See also TABLE I IN APPENDIX II "Principles of Electrical Engineering")

8.—Give an account of some practical application of "skin-effect" at low frequencies.

ANSWER.—From the theory of the three-phase induction motor it is known that for a high value of the starting-torque it is necessary to have a sufficiently large value of non-inductive resistance in the rotor circuit, whilst for normal running conditions it is essential that the rotor resistance shall be as low as possible in order to obtain high efficiency of operation.

These requirements are very satisfactorily met in the case of slip-ring motors by connecting starting resistances in series with the slip-rings so that when the rotor has reached full speed, the rotor starting-resistance is cut out and the slip-rings are short-circuited.

In the case of squirrel-cage rotors, however, this procedure is not possible since the rotor winding is permanently short-circuited and in consequence an inherent disadvantage of such motors is that they develop a relatively low torque at starting. During the last two decades an immense amount of research has been carried out with a view to improving the starting torque of squirrel-cage induction motors without interfering with its very satisfactory characteristics under normal running conditions.

A very successful device for obtaining the requisite high starting torque makes use of the principle of " skin effect " and this will be understood by reference to Figs. 11 and 12. In Fig. 11*a* is shown a relatively deep and narrow conductor in position in a slot of the rotor, the depth of the conductor being D cm. and the width of the slot t cm. If it be assumed that a current of 1 ampere is flowing in the conductor and that the current distribution is uniform over the whole cross-section of the conductor, a magnetic field will be established of which the magnitude can be calculated as follows.

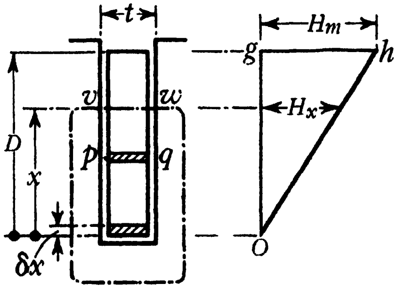
A typical line of magnetic force is shown by the chain-dotted path shown in Fig. 11*a* and if H_x oersted is the magnetic intensity in the slot viz., in the path $v\ w$, the current which will be linked with this line of force will be $\frac{x}{D}$ ampere since 1 ampere of current is passing in the whole conductor. Neglecting the reluctance of the path in the iron part of the magnetic circuit, the following magneto-motive force relationship will hold, viz.,

$$\frac{4\pi}{10} \times \frac{x}{D} = H_x \cdot t$$

or

$$H_x = \left(\frac{4\pi}{10D.t}\right)x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

from which it follows that the intensity H_x is a straight line function of x as shown in Fig. 11*b*.



(*a*) (*b*)

Fig. 11

Now a little consideration will show that an elementary strip of width δx at the base of the conductor as shown in Fig. 11*a* will be linked with the whole of the flux which crosses the slot whilst a strip such as pq, which is higher up in the slot, will be linked with a correspondingly smaller amount of flux.

Referring now to Fig. 12*a* in which the same conditions are shown as already described in the foregoing in connection with Fig. 11*a*, it is now more convenient to define the position of an elementary strip such as pq by its distance y from the top of the conductor, and similarly, in Fig. 12*b* the magnetic intensity H_y is defined with reference to the distance y from the top of the conductor. The magnetic flux which is now linked with the strip pq is given by the shaded area of the magnetic intensity triangle of Fig. 12*b* so that the general expression for the flux linked with the portion of the conductor which extends from the base of the conductor to the height $x = D - y$ is

$$\Phi = \int_0^y H_y dy = \int_0^y H_m \frac{D-y}{D} dy$$

that is,

$$\Phi = \frac{H_m}{D}\left[y.D - \tfrac{1}{2}y^2\right]$$

For example, if $y = 0{\cdot}2D$ then

$$\Phi = \frac{H_m}{D}\left[0{\cdot}2D^2 - 0{\cdot}02D^2\right]$$

In Fig. 12*c* the values of the flux as a function of the depth y are shown by the parabola curve

If the current in the conductor is alternating, then the flux curve of Fig. 12*c* will also be a measure of the e.m.f. which will be induced in the conductor at any depth y from the top of the conductor.

When the conductor shown in Figs. 11*a* and 12*a* forms a member of a squirrel-cage rotor, the rotating magnetic field will generate a uniform e.m.f. throughout the whole section and the consequent current will develop a back e.m.f. which will be a maximum at the base of the conductor and diminish towards the surface. The result of these two opposed e.m.f.s, viz., that due to the rotating field and that due to the flux-linkages of Fig. 12*b*, is that the current will be throttled in the lower part of the conductor section and forced into the upper part of the section. In other words, the effective

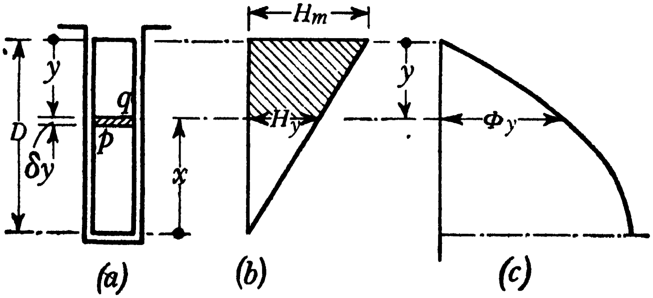


Fig. 12

resistance of the conductor to alternating current will be increased and this increase will be greater the higher the frequency of the alternating current. These conditions are exactly in accord with the starting requirements of a squirrel-cage induction motor. At starting, the frequency of the currents will be that of the supply mains (50 hz). When running with a normal slip of, say, $s = 2$ per cent., the frequency of the currents in the rotor conductor will only be $0{\cdot}02 \times 50 = 1$ hz.

It is estimated that up to date there are more than 100,000 squirrel-cage motors built with smooth deep strip conductors (e.g., section 6·5 cm. × 0·15 cm.). These comprise slow-speed motors for driving reciprocating compressors of 2,700 horse-power ; motors with a moment of inertia of many hundred ton-m^2 units: and motors for speeds of 3,000 and 1,500 r.p.m. for outputs up to 5,200 horse-power for driving turbo-blowers. All these motors are running in a perfectly satisfactory manner.

9.—Derive the conditions for which the damping of an iron core, alternating current inductance coil becomes a minimum.

ANSWER.—The damping due to inductance coils as used for low-frequency technical purposes and for high-frequency carrier-current telephony is almost always undesirable. The characteristic quantity which defines the damping is the " figure of quality," viz :

$$Q=\frac{\omega L}{R_{ef}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where ωL is the reactance of the coil and R_{ef} the total effective resistance of the coil. This effective resistance comprises (i) the d.c. resistance of the winding and supplementary losses such as those due to skin-effect and dielectric loss, and (ii) the iron core losses of the coil. In what follows it will be assumed that the supplementary losses of (i) are negligibly small in comparison with the other losses, so that,

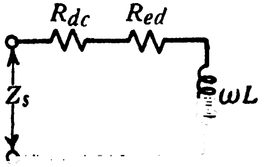


Fig. 13

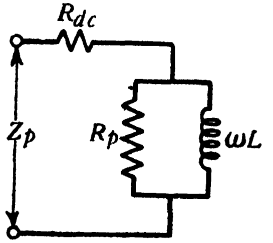


Fig. 14

$$R_{ef} = R_{dc} + R_{ed} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The hysteresis loss of the iron core will be neglected so that R_{ed} will be the resistance corresponding to the eddy current losses in the iron core, and these eddy current losses will be proportional to the square of the frequency.

The equivalent circuit can then be represented by the diagram of Fig. 13 where $\mathfrak{Z}_s$ defines the impedance of this circuit. In what follows, it will be assumed that the figure of quality is not too small, that is, $Q > 10$.

It is convenient to transform the series circuit of Fig. 13 into the equivalent series-parallel circuit of Fig. 14, the impedance of which is denoted by $\mathfrak{Z}_p$.

The value of the resistance R_p Fig. 14, which will be required in order that the series-parallel circuit shall be equivalent to the series circuit of Fig. 13 can then be found as follows:

The impedance of the series circuit is

$$\mathfrak{Z}_s = R_{d\,c} + R_{ed} + \mathrm{j}\ \omega L$$

and the impedance of the series-parallel circuit is

$$\mathfrak{Z}_p = R_{d.c} + \frac{1}{\dfrac{1}{R_p} + \dfrac{1}{j\omega L}}$$

so that for the required equivalence of the two circuits,

$$R_{ed} + \mathrm{j}\omega L = \frac{1}{\dfrac{1}{R_p} + \dfrac{1}{j\omega L}}$$

that is,

$$R_{ed} + \mathrm{j}\omega L = \frac{\mathrm{j}R_p \times \omega L}{R_p + \mathrm{j}\omega L}$$

and after rationalizing the expression on the right-hand side and equating respectively the real and the imaginary quantities, it is found that,

$$\left.\begin{aligned} R_{ed} &= \frac{R_p\,(\omega L)^2}{R_p^2 + (\omega L)^2} \ldots\ldots\ldots\ldots (a) \\ \omega L &= \omega L\left[\frac{R_p^2}{R_p^2 + (\omega L)^2}\right] \ldots\ (b) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (3)$$

An essential condition therefore is that the term in square brackets on the right-hand side of (*b*) must be $\simeq 1$ and this implies that R_p must be very large in comparison with (ωL).

The required value of R_p can now be found from the quadratic equation (3 *a*), viz.:

$$R_p^2 - \frac{R_p}{R_{ed}}(\omega L)^2 + (\omega L)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Of the two solutions of this equation that which gives the larger value for R_p is the only one applicable since it has already been seen that R_p must be relatively very large. The required solution is then found to be

$$R_p \simeq \frac{(\omega L)^2}{R_{ed}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and, since R_{ed} is proportional to the square of the frequency, it follows that R_p is also independent of the frequency.

The equation (1) may now be rewritten

$$Q = \frac{\omega L}{R_{eff}} = \frac{\omega L}{R_{dc} + R_{ed}} = \frac{\omega L}{R_{dc} + \frac{(\omega L)^2}{R_p}} \quad \ldots\ldots\ldots\ldots (6)$$

The damping will be a minimum when $\frac{dQ}{d\omega} = 0$

that is, when $\omega L = \sqrt{R_p \times R_{dc}}$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (7)

From equations (2) and (5) it is found that

$$R_{eff} = R_{dc} + R_{ed} = R_{dc} + \frac{(\omega L)^2}{R_p}$$

and substituting for R_p from (7) the condition for minimum damping becomes

$$R_{eff} = 2 R_{dc} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

that is to say [see equation (2)], the damping is a minimum when

$$R_{ed} = R_{dc}$$

or, otherwise expressed, for minimum damping, the *iron loss must be equal to the copper loss.*

10.—Give an account of the characteristic features of a "pot" type oscillator or resonator for ultra-high-frequency systems.

ANSWER.—For ultra-high-frequency circuits the structural design of the component parts must be of a special kind. Not only is it necessary to consider each connecting lead and terminal as a separate conductor, but it has also been found to be impracticable to build up satisfactory oscillatory circuits with the normal types of inductances and condensers. Progress in this direction has involved a departure from the conception of an inductance as comprising a coil of a number of turns in series. For ultra-high-frequency circuits, the parallel connection of such turns is essential, and the consequence of the application of this idea has been the development of the " oscillation pot."

Suppose, for example, that the simple series arrangement of a loop and a condenser as shown in Fig. 15 is rotated about the axis AB, the loop will then generate a closed cylindrical circuit as shown in Fig. 15, thus forming an " oscillation-pot." So long as the dimensions of this pot are small in comparison with the wavelength of the current oscillations, the calculation of the capacitance, inductance, and effective resistance can be made in a relatively simple manner. This characteristic feature is primarily due to the fact that the electric and magnetic fields are of a simple and definite form as a consequence of the screening effect of the walls of the pot. This structural form also implies the very great technical advantage that there are no stray fields. The magnetic lines of force are circular, and form a well-defined ring-shaped bundle. For the form of pot shown in Fig. 16, the coefficient of self-induction is given by the expression,

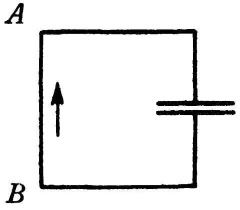


Fig. 15

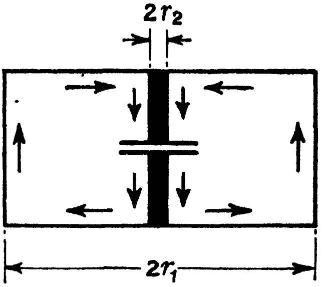


Fig. 16

$$L = 2l \log_e \frac{r_2}{r_1} \text{ elec. mag. units.}$$

where l cms. is the axial length $A\ B$

whilst the capacitance is given by the well-known formula for parallel plate condensers.

The amplitude of the current waves which penetrate the wall of the pot will decay rapidly, in accordance with the expression

$$e^{-2\pi x\sqrt{\chi . f}} \quad \text{(i)}$$

where,

x cm. is the depth of penetration,

χ electromagnetic units is the electrical conductivity per cm./cm.^{2} of the wall of the pot,

f hertz is the oscillation frequency.

Within the small distance

$$x_0 = \frac{1}{\sqrt{\chi . f}} \text{ cm.} \quad \text{(ii)}$$

the expression (i) becomes

$$e^{-2\pi} = e^{-6 \cdot 3} = 0 \cdot 0018$$

so that the oscillatory current will then have become completely suppressed.

Thus for a wavelength of 1 metre, i.e., $f=3\times10^8$ hertz, and for a copper pot, i.e., $\varkappa = 5{\cdot}9\times10^{-4}$ electromagnetic units per cm./cm.^{2},

$$\sqrt{\varkappa{\cdot}f} = \sqrt{(5{\cdot}9\times10^{-4})\times(3\times10^8)} = 4{\cdot}2\times10^2$$

and $$x_o = \frac{1}{\sqrt{\varkappa{\cdot}f}} = 0{\cdot}0024 \text{ cm.} = 0{\cdot}024 \text{ mm.}$$

11.—A copper oscillation-pot of the form described in Example 10, carries an oscillatory current of 1 metre wavelength. Find the effective resistance and the power loss per sq. cm. of the cylindrical wall. The specific resistance of copper is $\rho=1{\cdot}77\times10^{-6}$ ohm per cm. cube.

ANSWER.—From the expression (49), page 452 of " Principles of Electrical Engineering," the equivalent depth of penetration of the current is

$$\sigma = \frac{1}{2\pi}\sqrt{\frac{\rho.10^9}{\mu.f}} = 5{,}030\sqrt{\frac{\rho}{\mu.f}} \text{ cm.}$$

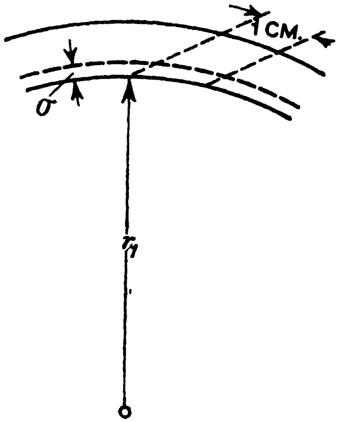


Fig. 17

For a wavelength of 1 metre, the corresponding frequency is $f = 3\times10^8$ hertz and for copper, the permeability is $\mu = 1$. Hence,

$$\sigma = 5{,}030\sqrt{\frac{1.77\times10^{6}}{3\times10^8}} = 0.000387 \text{ cm.} = 0.00387 \text{ mm.}$$

The effective resistance per sq. cm. of surface area of the cylindrical wall is then,

$$R_1 = \rho\,\frac{1}{\sigma} = \frac{1.77\times10^{-6}}{0.000387} = 4.55\times10^{-3} \text{ ohm.}$$

The power loss per sq. cm. area of the cylindrical wall is then,

$$W_1 = (I_{rms})^2 R_1 \text{ watts} = (I_{rms})^2\times4.55\times10^{-3} \text{ watts} \quad \ldots\ldots\ldots\text{(i)}$$

where I_{rms} ampere is the r.m.s. value of the current per cm. of cylindrical periphery of the wall of the pot (see Fig. 17). If $(I_{r.m.s})_T$ ampere is the r.m.s. value of the total current in the cylindrical wall, then the value of I_{rms} in the expression (i) is

$$(I_{rms}) = \frac{(I_{rms})_T}{2\pi r_1} \text{ ampere}$$

as shown in Fig. 17.

12.—*Discuss the significance of the oscillation-pot with respect to its " resonance quality" $Q_{res.}$ and its " resonance resistance " $R_{res.}$*

Answer.—Since the current in an oscillation-pot resonator can spread over a large conducting surface and since the current heating losses are extremely small, an extraordinarily high value for the resonance peak is obtained in spite of the unfavourable value of the ratio $\frac{L}{C}$. That is to say, if R_W is the resistance corresponding to the losses (as found in Example 11), then

$$Q_{res} = \frac{1}{R_W} Z_0 = \frac{1}{R_W}\sqrt{\frac{L}{C}}$$

and has an extremely high value, varying from about 10,000 to 50,000 according to the material and shape of the pot structure. The resonance quality thus reaches values which are many times greater than can possibly be obtained with a normal oscillatory circuit (see also, expression (5), page 259, Chapter IX).

The " resonance resistance " is then

$$R_{res} = Q_{res}\, Z_0 = \frac{1}{R_W}\,\frac{L}{C}$$

In general, however, it is not practicable to take full advantage of the high resonance quality Q_{res} or of the high resonance resistance R_{res} because when the oscillation-pot is used as an element of a circuit, it takes energy out of the circuit and frequently gives it up to another circuit. The damping is then determined by the consequent reaction effect and the pot operates as a " no-loss " transformer.

Similarity relationships can be obtained as follows : For a proportional change in the linear dimensions, the natural wavelength of the oscillation-pot changes in the same proportion whilst the " loss resistance " R_W changes inversely as the square-root of the change of linear dimensions.

For oscillating currents of a wavelength greater than a few metres, the oscillation-pot can no longer be used since its dimensions become too large. On the other hand, it is an admirable circuit component for the very shortest wavelengths. Summarising its leading characteristic features, it may be said that they are : favourable dimensions ; high resonance quality ; high resonance resistance ; simplicity of the calculations with regard to all its parts ; perfect screening ; ease of coupling to other circuits.

Chapter XV

PROPAGATION OF ELECTRIC ENERGY ALONG TRANSMISSION LINES AND CABLES

1.—If a three-phase transmission line is " compensated " by connecting a uniformly distributed reactance between each line and the star point, find what reactive volt-amperes will be taken by this reactance for any given value of the load at the consumer's terminals.

ANSWER.—When a transmission line is carrying its natural load, the pressure and the current respectively have the same magnitudes at every point of the line (see page 478 of " Principles "). It has also been seen that if, for example, the consumer's load is less than the natural power, the pressure at the consumer's terminals will rise, unless special means are provided to prevent it. The natural power is defined by the expressions,

$$W_n = 3\frac{V_p^2}{Z_o} = 3V_p^2\sqrt{\frac{C}{L}} = V_l^2\sqrt{\frac{C}{L}}\text{ watts} \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

where V_p volts is the phase pressure, V_l volts the line pressure, and Z_o ohms the surge impedance per phase of the line. When the load is equal to the natural power, then in the element of the circuit as shown in Fig. 1, the pressure rise in the inductance L due to the capacitative current will be just equal to the pressure drop in L due to the load current I_L, so that the line pressure remains constant throughout.

When the load is less than the natural power, the pressure rise in L due to the capacitance current I_c will be greater than the pressure drop due to the load current I_L, so that there will be a net *pressure rise* which will be cumulative from the generator end to the consumer's end. If, however, the capacitance C were to be reduced when the load is less than the natural load, the pressure rise due to the capacitance current in L would diminish, so that it would be possible to obtain constant line pressure by adjusting the magnitude of the capacitance to suit the magnitude of the load at the consumer's terminals, and so maintain constant pressure throughout the line.

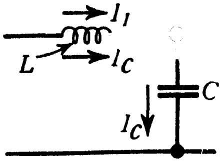


Fig. 1

One method by which such an adjustment of the line capacitance could be made would be to connect a uniformly distributed inductance λ henry per unit length of line in parallel with the capacitance as shown in the element of the circuit in Fig. 2, the effect of such an inductance being to reduce the effective capacitance across the line. For example, if $\omega C = \frac{1}{\omega\lambda}$ then a condition of parallel resonance is obtained. For any given value of the inductance λ, the effective uniformly distributed admittance across the line will be, $j\omega C + \frac{1}{j\omega\lambda} = j\omega\left(C - \frac{1}{\omega^2\lambda}\right) = j\omega(C - K)$ per unit length of line so that the effective distributed capacitance across the line will be $C - \frac{1}{\omega^2\lambda} = (C - K)$ farad per unit length of the line where $K = \frac{1}{\omega^2\lambda}$ farad per unit length of the line, and the power transmitted by the line to the consumer will then be [see (i)], $W_L = 3\,V_P{}^2\sqrt{\frac{C-K}{L}} = V_l{}^2\sqrt{\frac{C-K}{L}}$ watts .... (ii)

Dividing (ii) by (i) gives, $\frac{W_L}{W_n} = \sqrt{\frac{C-K}{L}}\sqrt{\frac{L}{C}} = \sqrt{1-\frac{K}{C}}$

so that $$\left(\frac{W_L}{W_n}\right)^2 = 1 - \frac{K}{C} \quad \text{..................(iii)}$$

But $\frac{K}{C} = \frac{3\,V^2\,\omega\,K}{3\,V^2\,\omega\,C} = \frac{W_K}{W_C}$ where W_K is the extra reactive power which passes across the line due to the added inductance λ, and W_C is the reactive power taken by the capacitance C. Again, $\frac{W_c}{W_n} = \frac{I_c}{I_n}$ where $I_c = V_P\omega C.s$ and is the total current per phase taken by the capacitance, $I_n = \frac{V_P}{Z_0} = V_P\sqrt{\frac{C}{L}}$ is the natural load current per phase, and s the transmission distance so that, $\frac{W_c}{W_n} = \omega C.s\sqrt{\frac{L}{C}} = \omega.s\sqrt{LC} = \frac{\omega.s}{c}$

$$\text{and}\quad \frac{W_K}{W_n} = \frac{W_K}{W_c} \times \frac{W_c}{W_n} = \frac{K}{C}\,\frac{\omega.s}{c}$$

$$\text{and since from (iii)}\quad \frac{K}{C} = 1 - \left(\frac{W_L}{W_n}\right)^2 = \frac{c}{\omega.s}\,\frac{W_K}{W_n}$$

$$\text{therefore}\quad \frac{W_K}{W_n} = \frac{\omega.s}{c}\left\{1 - \left(\frac{W_L}{W_n}\right)^2\right\} \quad \text{.............. (iv)}$$

So long as the load W_L is less than W_n this expression is positive and the requisite extra reactive power is *inductive.* When the load is greater than the natural power this expression becomes negative, which means that the extra reactive power must be capacitative, and instead of an inductance λ in Fig. 2 a capacitance must be connected.

If $\frac{W_K}{W_n}$ be denoted by y and $\frac{W_L}{W_n}$ be x, the expression (iv) becomes

$$y = \frac{\omega.s}{c}\left\{1 - x^2\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$

This equation represents a parabola with the apex on the ordinate axis as shown in Fig. 3. It is to be observed that there is no theoretical limit to the amount of power which can be transmitted under these conditions, viz., when the line resistance is negligible and when the compensating reactive power is fed into the line by means of a uniformly distributed reactive appliance. Nevertheless, when the load is greater than about 1·5 W_n the necessary capacitative reactive power becomes prohibitively large.

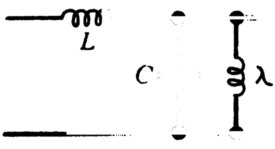


Fig. 2

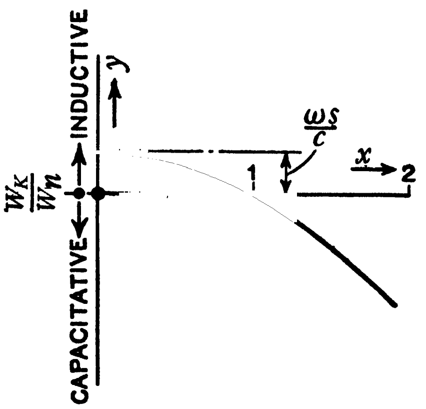


Fig. 3

2.—*If the line pressure at each end of an unloaded three-phase transmission line is maintained at the constant value of 132 kV., find: (i) What capacitance current must be supplied to each end of the line; (ii) what pressure rise will develop at the mid-point of the line. The surge-impedance is $Z_0 = 380$ ohms per phase, the supply frequency is 50 and the length of the line is $2s = 200$ km.*

ANSWER.—In Fig. 4 is shown one phase of the line to one end of which the generator is connected, and to the other end B some appliance is connected which can supply the necessary capacitance current to maintain the pressure at B equal to that at G.

$$\text{Then } Z_0 = \sqrt{\frac{L}{C}} : \sqrt{LC} = \frac{1}{c} : \text{ where } c = 3 \times 10^5 \text{km./sec.}$$

$$\text{so that } C = \frac{1}{Z_0 c} = \frac{1}{380 \times 3 \times 10^5} \text{ farad/phase/km.}$$

$$\text{and } L = Z_0^2 C = \frac{380}{3 \times 10^5} \text{ henry/phase/km.}$$

The capacitance current, which must be supplied at each end of the line, is then $I_c = V_p \omega C s = \frac{130{,}000}{\sqrt{3}} \times \frac{2\pi \times 50 \times 100}{380 \times 3 \times 10^5} = 20{\cdot}7$ amperes per phase.

This current will be distributed along the line as shown in Fig. 5, so that at a point distant x km. from either end $\frac{I_x}{I_c} = \frac{s-x}{s}$, where $I_c = 20{\cdot}7$ amperes. The pressure rise at any point distant x km. from either end will be $\triangle V = \int_0^x \omega L I_x dx = \omega L I_c \left[x - \frac{1}{2}\frac{x^2}{s}\right]$ volts .... (i)

and for the middle of the line, i.e., for $x = s$:

$\triangle V = \frac{1}{2} I_c \omega L s = 407$ volts per phase $= 710$ line volts as shown in Fig. 6.

It is to be noted that for the pressure rise at the mid-point of the line

$\triangle V = \omega L$ (shaded area of Fig. 5) volts/phase

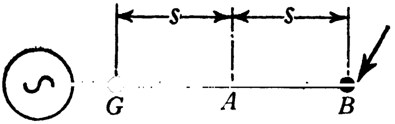


Fig. 4

Fig. 5

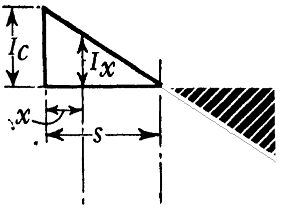
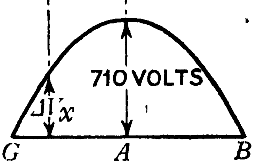


Fig. 6



$$\text{and } \frac{\triangle V}{V} = \frac{710}{132{,}000} 100 = 5{\cdot}5\%$$

The graph for the pressure rise at any point on the line as defined by the expression (i) is a parabola (Fig. 6).

3.—Referring to Example 2, one method by means of which the necessary capacitance current can be supplied to the receiver's terminals is to connect at that end of the line inductance of value L_B henry, such that it will draw a current of 20.7 amperes from that end of the line. Explain how it is that an inductance can be said to supply a capacitance current to the line in this way.

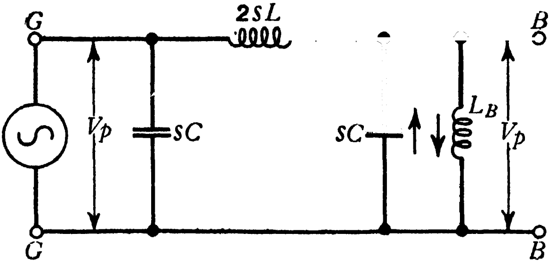


Fig. 7

ANSWER.—The physical significance of this method of supplying a capacitance current to the line can be seen most easily if the line is replaced by its equivalent 4-pole π system as has been explained on page 484. That is to say, the π system as shown in Fig. 7, in which one-half of the total capacitance $2{\cdot}s{\cdot}C$ is concentrated at each end of the line, and the whole of the inductance $2{\cdot}s{\cdot}L$ is concentrated in between the two lumped capacitances. Thus, at each end of the line, a capacitance $s{\cdot}C$ farad is concentrated and in between a concentrated inductance of $2{\cdot}s{\cdot}L$ henry.

If now the inductance L_B which is connected to the end B of the line is such that the current flowing in it is $I_L = \frac{V_P}{\omega L_B} = I_c = V_P\,\omega\,sC = 20{\cdot}7$ amperes then $\omega^2 L_B\,(sC) = 1$, so that the parallel arrangement of sC and L_B is in resonance at the supply frequency. The capacitance current is, however, 180 out of phase with the current in the inductance L_B, so that the current circulates round this closed loop. That is to say, an inductive current *drawn* from the line is, in effect, the same as a capacitance current *fed into* the line.

If instead of the inductance L_B an unloaded synchronous motor were connected across the line at B then this machine would have to be " under-excited " (see Example 3, Chapter XII) in order to enable it to draw an inductive current from the line. At the generator end of the line the capacitance current is supplied directly by the generator, and such a capacitance current produces by its armature reaction a magnetising effect on the main field system, and thus tends to cause the pressure to rise. In order to maintain the pressure constant, therefore, at the terminals G, Fig. 7, the field excitation of the generator would have to be reduced.

4.—What is the " Lecher Line," and what is its purpose ?

ANSWER.—If two parallel wires are connected to a source of a.c., waves of pressure and current will be produced on the wires, and these waves will travel from the generator to the end of the line, and will then be reflected back to the generator. In this way *standing waves* of pressure and current will become established on the line, and their existence can be demonstrated, for example, by bridging the wires with a neon lamp. If such a bridge is moved along the wire, successive positions can be found at which the lamp will glow with maximum brightness. Such positions mark the anti-nodes of the standing wave of pressure, and the distance between two neighbouring anti-nodes is one-half wavelength, viz., $\frac{\lambda}{2}$, as shown in Fig. 8.

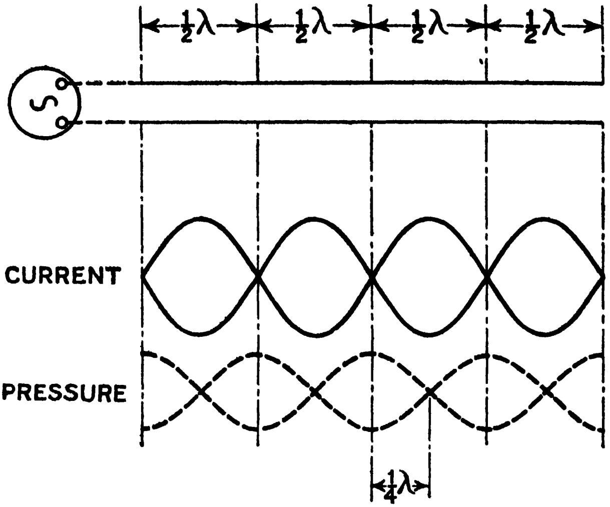


Fig. 8

The standing waves of current will be displaced by one-quarter wavelength with regard to those of pressure (see also page 479 of " Principles of Electrical Engineering "), and their existence can be demonstrated by bridging the wires with a small glow lamp and moving it along the wire. When the bridge reaches a current anti-node, the lamp will glow with maximum brightness, and the distance between two such neighbouring positions will be $\frac{\lambda}{2}$, as shown in Fig. 8. An example of the practical application of such a Lecher line is the tuned feeder system referred to in Example 2, Test Paper, Chapter XVI.

If a Lecher line is placed in a medium of which the dielectric constant is ε, for example, in a liquid or laid in the earth, the length of the standing waves will be a measure of the dielectric constant. Thus, for air, $\varepsilon = 1$, and if the wavelength in air is λ and the wavelength in a medium of dielectric constant ε^1 is λ^1, then : $\lambda^1 = \sqrt{\varepsilon^1}$, this relationship being in accordance with the Maxwell equations (ChapterXVI, page 494, "Principles").

(*Note.*—Lecher was a German scientist who demonstrated such standing waves in 1890.)

5.—Find what reactive kVA. per kilometre must be supplied to a three-phase overhead transmission line and show the results as a function of the line pressure for the following loads: (i) No load, (ii) 0.75 times the natural load, (iii) 1.25 times the natural load. The line constants are as follows:

Capacitance $C_0 = 0.0096 \times 10^{-6}$ farad per kilometre;
Inductance $L_0 = 0.00127$ henry per kilometre.
The supply frequency is 50 hz.

ANSWER.—This problem is most conveniently solved by reference to the equivalent 4-pole π unit of Fig. 9. From what has been explained already in Example 3 of this Test Paper, it will be seen that at no load the capacitance current which must be supplied at the consumer's end of the line can be obtained by connecting an inductance of L_x henry per phase across the consumer's terminals, as shown in Fig. 9, where,

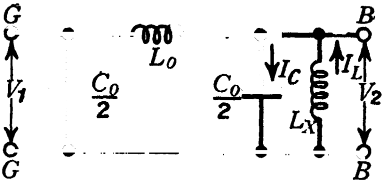


Fig. 9

$$\omega L_x = \frac{1}{\omega \frac{C_0}{2}}, \text{ so that } I_x = I_c = V_p \frac{\omega C_0}{2}$$

If the line pressure is 150 kV, then the capacitance current will be

$$I_c = \frac{150{,}000}{\sqrt{3}} \omega \frac{C_0}{2} = 87{,}000 \times 314 \times 0{\cdot}0048 \times 10^{-6} \text{ amperes}$$

$$= 0{\cdot}131 \text{ ampere per phase per km.}$$

The total reactive power which must be supplied at the consumer's end for the three phases will then be :—

$$3\,V_pI_c = 3\,V_pI_x = \{3(87{,}000)^2 \times 314 \times 0{\cdot}0048 \times 10^{-9}\} \text{ kVA} = \underline{34{\cdot}2\text{kVA}}$$

An equal amount of capacitative reactive power must be supplied at the generator end of the line so that the total reactive power to be supplied to the line when operating at 150 kV. and with no load on the consumer's terminals is 68·4 kVA, and since $\omega L_xI_L = 87{,}000$ volts, the value of the inductance will be $L_x = \frac{87{,}000}{314 \times 0{\cdot}131} = 2{,}110$ henry/phase/km.

The amount of kVA. necessary at no load for any other line pressure is obtained by reducing the above quantity in the ratio of the square of the line pressures, and in this way the parabola curve marked " no load " in Fig. 10 has been obtained.

The surge impedance of the line is $Z_o = \sqrt{\frac{L_o}{C_o}} = 363$ ohms per phase so that the natural power of the line is $\frac{V_l^2}{363}$ watts, which for a line pressure of $V_l = 150$ kV. gives $Wn = \frac{(150)^2 \times 10^6}{363} = 62 \times 10^6$ watts $= 62$ MW and the current for this natural power is $I_n = \frac{62 \times 10^6}{3 \times 87{,}000} = 250$ amperes per phase.

When the transmitted power W is less than W_n the effective capacitance C_o per km. must be reduced to a value C_o', where $Z_o' = \sqrt{\frac{L_o}{C_o'}}$ and $I = \frac{V_p}{Z_o'}$. As has been explained already in Examples 2 and 3 of this Test Paper, this can be done by connecting an inductance of λ henry per km. so that the necessary inductance to be connected across the consumer's terminals of the 4-pole π unit of Fig. 9 will be L_x henry where $\frac{WC_o'}{2} = \frac{WC_o}{2} - \frac{1}{\omega L_x}$. Thus, for $W = 0{\cdot}75\ W_n$: $\frac{V_l^2}{Z_o'} = 0{\cdot}75\ W_n = 46{\cdot}5$ MW and consequently, $Z_o' = \frac{62}{46{\cdot}5} Z_o = 1{\cdot}3 \times 363 = 471$ ohms per phase and hence $C_o' = \frac{L_o}{471^2} = \left(\frac{363}{471}\right)^2 C_o = 0{\cdot}59 C_o$ that is, $L_x = \frac{2}{\omega^2(C_o - C_o')} = \frac{2}{314^2 \times 0{\cdot}31 C_o}$

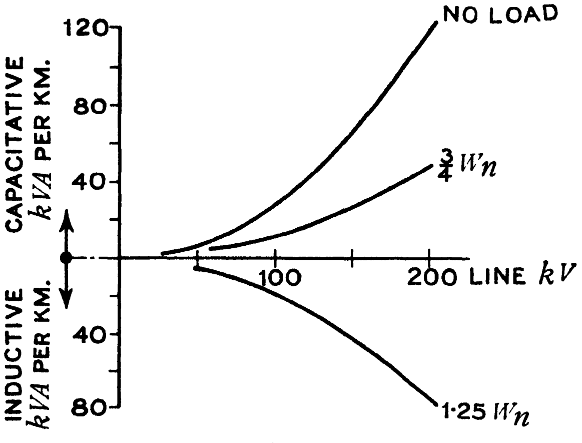


Fig. 10

$= \dfrac{2 \times 10^6}{9{\cdot}9 \times 10^4 \times 0{\cdot}41 \times 0{\cdot}0096} = \underline{6{,}150 \text{ henry}}$. The reactive power at the consumer's end for the three phases will be

$$3V_pI_L = \frac{3\,V_P{}^2}{\omega L_x} = \frac{3(87{,}000)^2}{314 \times 6{,}150} \times 10^{-3}\,\text{kVA} = \underline{11{\cdot}8 \text{ kVA.}}$$

An equal amount of reactive kVA must be supplied at the generator end of the line so that the total reactive power to be supplied when transmitting a load $W = 0{\cdot}75\ W_n$ at 150 kV line pressure is 23·6 kVA per km.

For other voltages the amount will be in the ratio of the square of the line pressures and in this way the curve of Fig. 10 is obtained.

When the load is greater than the natural load, the equivalent capacitance C_0' is greater than C_0 and hence L_x changes sign. In other words, a capacitance must now be connected across the consumer's terminals of an amount $C_0' - C_0$. This means that the consumer's end of the line must be *supplied* with inductive kVA. which is the same thing as saying that capacitative kVA. must be *taken from* the consumer's end of the line.

In Fig. 10 is shown the graph for the condition that the load is $1{\cdot}25\ W_n$.

The foregoing investigation refers to an overhead line. For an underground cable the line constants will be widely different from those of an overhead line, thus :

The velocity of propagation in a high-tension cable is about

$$c = 1{\cdot}5 \times 10^5 \text{ km./sec. so that, } \alpha = \frac{\omega}{c} = 2{\cdot}1 \times 10^{-3} \text{ radian per km.}$$

Representative values of the constants for such a cable are:

$L_0 = 0{\cdot}33 \times 10^{-3}$ henry per km. : $C_0 = 0{\cdot}133 \times 10^{-6}$ farad per km.

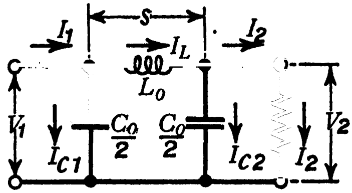
$\omega L_0 = 0{\cdot}11$ ohm per km. : $\omega C_0 = 0{\cdot}042 \times 10^{-3}$ siemens per km. and

$Z_0 = \sqrt{\dfrac{L_0}{C_0}} = 50$ ohms per phase, so that for a given pressure, the natural power of a cable is about 7 times that of an overhead line.

For all cases, the total reactive power which the three-phase line requires is $\dfrac{3}{10^3}\left\{ V_P{}^2\,\omega C_0 - I_2{}^2\,\omega L_0\right]\,s$ kVA. for s km. $= \dfrac{1}{10^3}\left\{ V_l{}^2\omega C_0 - 3I_2{}^2\omega L_0\right]\,s$ kVA. for s km: where V_l is the line pressure in volts and the load current I_2 is in amperes at $\cos\varphi = 1$.

6.—Draw the vector diagram for one phase of a three-phase transmission line by making use of the equivalent 4-terminal π unit. From this diagram derive the relationship which defines the natural power of the line.

ANSWER.—When the line is carrying its natural load, it is known from the results obtained on page 478 of " Principles " that the pressure is the same at each end of the line and that each of these pressures is in phase with the corresponding current in the line. The vector diagram for the natural load current I_L will therefore be as shown in Fig. 11. Inspection of this vector diagram shows that the following relationships hold, viz:—



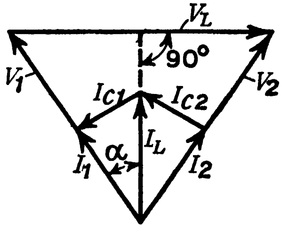


Fig. 11

$$V_L = 2V_1 \sin\alpha : I_{c_1} = V_1 \frac{\omega C_0}{2}$$

$$V_L = \omega L_0 I_L : I_L = \frac{I_{c_1}}{\sin\alpha}$$

so that, $V_L I_L = 2 V_1 I_L \sin\alpha = 2V_1 I_{c_1} = V_1^2 \omega C_0$

that is, $\omega L_0 \times I_L^2 = V_1^2 \times \omega C_0$:

$$I_L^2 = V_1^2 \frac{C_0}{L_0} \text{ or } I_L = \frac{V_1}{Z_0} \text{ where } Z_0 = \sqrt{\frac{L_0}{C_0}}$$

and is the surge impedance of the line.

7.—Show how the resistance of a transmission line may be taken into account in the equivalent 4-terminal π unit.

ANSWER.—The exact mathematical theory of transmission lines shows that for lines up to about 300–400 km. in length the 4-pole unit π system shown in Fig. 12 gives a practically exact equivalent per phase of the transmission line in so far as the input and output relationships are concerned, so that a study of the performance of such transmission lines can be accurately followed by means of the vector diagram derived from the system of Fig. 12.

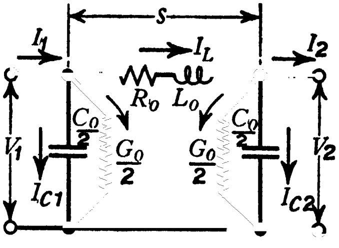


Fig. 12

In this diagram :

s km. is the length of the line.

$L_0 = sL$ henry is the inductance per phase of the line.

$C_0 = sC$ farad is the capacitance per phase of the line.

$R_0 = sR$ ohm is the resistance per phase of the line.

$G_0 = \frac{1}{r_0 s}$ siemens is the conductance per phase of the line, that is, the conductance of the line insulation, where r_0 is the insulation resistance per phase, per km. of the line. The quantities L, C, and R are the values of the line constants per phase, per km.

8.—Explain how an arc to earth which starts on an overhead line may be automatically suppressed (Petersen coil).

ANSWER.—The Petersen coil is an air-core inductance connected between the neutral point of the system and the earth. When the insulation of the system is sound and the load is symmetrical, no appreciable current will flow in this coil. The automatic action of the coil in suppressing an arc to earth across a line insulator can be seen by reference to the diagram of connections and the vector diagrams in Figs. 13, 14 and 15, where C farad per phase is the capacitance of the line to earth, and L henry is the inductance of the Petersen coil. If one of the lines, say S, arcs to earth across a line insulator the capacitance of that line becomes short-circuited, and the conditions will then be as shown in Fig. 14.

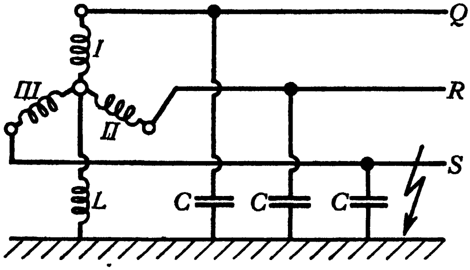


Fig. 13

Referring to the vector diagrams of Fig. 15 the three-phase pressure vectors with respect to the starpoint O are shown in Fig. 15*a*. In Fig. 15*b* the pressure of the line R with respect to the earthed line S is shown by V_{RS} of magnitude $\sqrt{3}V_{\text{RO}}$, and the pressure of the line P with respect to S is shown by V_{QS}. The capacitance current in the line R is shown by I_{R} in Fig. 15*b*. The magnitude of this current being

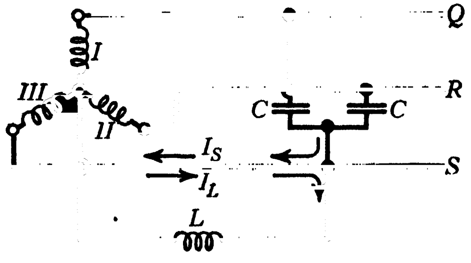


Fig. 14

$$I_{\text{R}} = \sqrt{3}\ V_{\text{P}}\omega C \text{ amperes.}$$

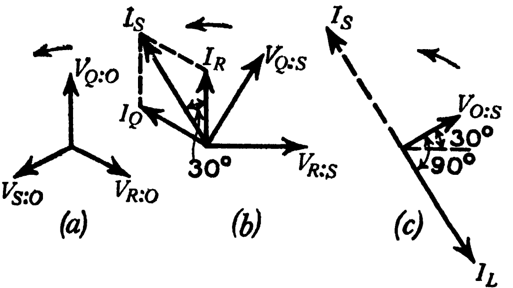


Fig. 15

The capacitance current in the line Q is shown by I_{Q} and the magnitude of this current is also $I_{\text{Q}} = \sqrt{3}\ V_{\text{P}}\omega C$

The capacitance current which will flow in the earthed line s will then be given by the vector sum of I_{R} and I_{Q} will be $I_{\text{S}} = \sqrt{3}I_{\text{Q}} = 3V_{\text{P}}\omega C$.

There will also flow in the line S a current I_L due to the pressure of the neutral point O with respect to the line S, and this current is shown in Fig. 15*c*, lagging by 90° on the pressure vector V_{OS}. If then the inductance L has such a magnitude that $I_L = I_S$ then these two currents will neutralise each other (see Fig. 14) so that no current will flow through the point of short-circuit, that is to say, the arc will be quenched. The requisite condition for this result is that $I_L = I_S$ that is $\frac{V_P}{\omega L} = 3V_P\omega C$ or $\frac{1}{\omega L} = \omega(3C)$ and this is seen to be also the condition that, at the supply frequency, the inductance L shall be in resonance with the three line capacitances in parallel.

9.—*Define the " bel " : " decibel " : " neper." From these definitions derive the numerical relationships between the three quantities.*

ANSWER.—The power magnitudes W_1 and W_2 are said to differ by N bels when

$$\frac{W_1}{W_2} = 10^N$$

that is

$$N = \log_{10} \frac{W_1}{W_2} \text{ bels.}$$

If $W_1 = W_2$, then $N = 0$. If N is in *decibels*, then

$$\frac{W_1}{W_2} = 10^{0 \cdot 1N}$$

so that

$$N = 10 \log_{10} \frac{W_1}{W_2} \text{ decibels}$$

The " neper " is such that two power magnitudes W_1 and W_2 are said to differ by N nepers when $\frac{W_1}{W_2} = (e^2)^N$

so that, $\frac{W_1}{W_2} = e^{2N} : N = \frac{1}{2} \log_e \frac{W_1}{W_2}$ nepers,

Thus, for example, 60 decibels = 6 bels = $\log_{10} \frac{W_1}{W_2}$

so that

$$\frac{W_1}{W_2} = 10^6$$

$$N = \tfrac{1}{2} \log_e 10^6 = \tfrac{1}{2} \times 2{\cdot}303 \times 6 = 6{\cdot}909 \text{ nepers}$$

that is

6 bels = 6·909 nepers
1 bel = 1·15 nepers

The decibel is 1/10th part of a " bel " (so called after A. Graham Bell, the inventor of the telephone) and by means of this unit, power ratios, and gain or loss ratios of related quantities such as currents and voltages, can be simply expressed. It originated in line telephony technique in 1923, when the American Telephone and Telegraph Company introduced a new unit, then called a " transmission unit." This was intended to replace an older method of comparison based on a ratio between the decrease in signal strength produced by the given telephone line and that produced by a " mile of standard cable." In 1924, an International Advisory Committee on long-distance telephony in Europe, together with the representatives of the Bell system, agreed to recommend their countries to adopt as standards the following, viz. :—

Either, the " bel," a unit involving logarithms to the base 10, and equal to 10 of the American Company's " transmission units,"

Or, the " neper " (after Napier, the inventor of logarithms), a unit involving logarithms to the base e.

One advantage of the decibel and neper as methods of expressing power, voltage and current ratios is that the enormous range of these ratios in communication work can be expressed in numerical values which are conveniently small, instead of the immensely cumbersome expressions which would be necessary otherwise.

Suppose, for example, that the strength of a cable signal obtained at the receiving end is 1/10th that at the transmitting end, the loss is then 1 bel. With two similar cables in series the loss would be 2 bels.

If the output power of an amplifier is 100 times the input, the " gain " is 2 bels or 20 decibels. With two such amplifiers in series the ratio of the output to input power would be 10,000 : 1, that is, 4 bels or 40 decibels. It is thus to be noted that the net power ratio of two amplifiers in series involves the *product* of their individual power ratios, whereas the decibel representation is given by the sum of the two individual corresponding quantities.

An *attenuator* may be regarded as the converse of an amplifier, and its power loss may be described in similar units. If two stages of amplification are followed by a line having an attenuation of s decibels, the net power level at the output end of the system will be $(p + q - s)$ decibels above the original power level at the input.

In general, the two powers P_1 and P_2 will be compared by observing *either* the voltage developed across a given impedance *or* the current through it. If the input and output impedances of (say) an amplifier are equal, the power ratio will be proportional to the square of the voltage or the current ratio, viz. :—

$$\frac{P_1}{P_2} = \left(\frac{V_1}{V^2}\right)^2 = \left(\frac{I_1}{I_2}\right)^2$$

so that, $$N = 10 \log_{10} \left(\frac{V_1}{V_2}\right)^2 = 20 \log_{10} \frac{V_1}{V_2} \text{ decibels,}$$

or $$N = 20 \log_{10} \frac{I_1}{I_2} \text{ decibels} = 2 \log_{10} \frac{I_1}{I_2} \text{ bels,}$$

also $$N = \tfrac{1}{2} \log_e \left(\frac{V_1}{V_2}\right)^2 = \log_e \frac{V_1}{V_2} = 2{\cdot}3 \log_{10} \frac{V_1}{V_2} \text{ nepers.}$$

If the attenuation or amplification in decibels (or bels) is known, the corresponding power and voltage (or current) ratios can be found by means of a table of antilogarithms, as will be seen from the following numerical examples :—

Decibels	Bels	Power Ratio	Voltage Ratio
0.1	0.01	$\frac{W_1}{W_2} = 10^{0\cdot01}$ = antilog of 0.01 = 1.023 :	$2 \log_{10} \frac{V_1}{V_2} = 0.01$: Antilog of 0.005 = 1.012
0.2	0.02	$\frac{W_1}{W_2} = 10^{0\cdot02}$ = antilog 0.02 = 1.047	$2 \log_{10} \frac{V_1}{V_2} = 0.02$: Antilog 0.01 = 1.03
0.3	0.03	$\frac{W_1}{W_2} = 10^{0\cdot03}$ = antilog 0.03 = 1.072	$2 \log_{10} \cdot \frac{V_1}{V_2} = 0.03$ Antilog of 0.015 = 1.035
10	1	$\frac{W_1}{W_2} = 10^1$ = antilog of 1 = 10	$2 \log_{10} \frac{V_1}{V_2} = 1$ Antilog of 0.5 = 3.162

For the measurement of relative loudness, the unit termed "phon" has been introduced. The arbitrary datum level of loudness is the r.m.s. pressure, at the threshold of audibility, of a free progressive tone at 1,000 cycles per sec. This r.m.s. pressure has been standardised in this country at 0·0002 dynes per sq. cm. More accurately, it is an intensity level of 10^{-16} watts per sq. cm., corresponding to a sound pressure of 0·000204 dynes per cm.^{2}, at 76 cm. mercury pressure and 20° C. Loudness may be expressed as a ratio above this arbitrary datum. The loudness of a sound in *phons* is numerically equal to the sound intensity in decibels of an equally loud, 1,000 frequency, pure note.

10.—A resistance of R ohms connects an underground cable of surge impedance $Z_{02} = 50$ ohms with an overhead line of surge impedance $Z_{01} = 500$ ohms. The resistance is of such a magnitude that it gives maximum efficiency of absorption of energy. Construct the graphical representations of the current and pressure for the following conditions, viz.: (i) When the surge passes from the overhead line into the cable; (ii) when the surge passes from the cable into the overhead line. Contrast these results with those obtained when the resistance $R = 0$.

Answer.—The general equations for these conditions are:

$$\dot{i}_{f_2} = \dot{i}_{f_1} \frac{2Z_{01}}{Z_{01} + Z_{02} + R} : v_{f_2} = v_{f_1} \frac{2Z_{02}}{Z_{01} + Z_{02} + R}$$

$$\dot{i}_{r_1} = \dot{i}_{f_1} \frac{Z_{01} - Z_{02} - R}{Z_{01} + Z_{02} + R} : v_{r_1} = v_{f_1} \frac{R + Z_{02} - Z_{01}}{Z_{01} + Z_{02} + R}$$

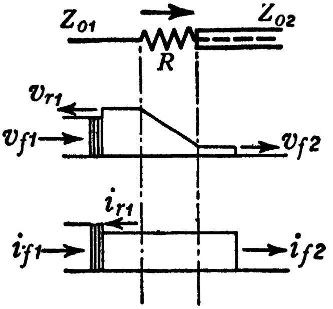


Fig. 16

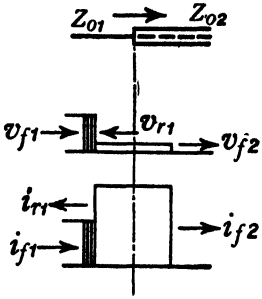


Fig. 17

The power absorbed by the resistance R will be $R\,(\dot{i}_{f_2})^2$ watts, that is,

$$R\left(\dot{i}_{f_1} \cdot \frac{2Z_{o_1}}{Z_{o_1} + Z_{o_2} + R}\right)^2 \text{ watts.}$$

For a given value of the current $\dot{i}_{f_1}$ the power absorbed will be a maximum when

$$\frac{d}{dR}\left[\frac{R}{(Z_{o_1} + Z_{o_2} + R)^2}\right] = 0$$

from which it follows that the efficiency of absorption of the resistance R is a maximum when

$$R = Z_{o_1} + Z_{o_2}$$

For $Z_{01} = 500$ ohms: $Z_{02} = 50$ ohms: and, for maximum efficiency of absorption $R = Z_{01} + Z_{02} = 550$ ohms.

Case (*i*) $R = 550$ ohms: Fig. 16 ·

$$\dot{i}_{f_2} = \dot{i}_{f_1} \frac{1{,}000}{1{,}100} = 0{\cdot}91\, \dot{i}_{f_1} : v_{f_2} = v_{f_1} \frac{100}{1{,}100} = 0{\cdot}091\, v_{f_1}$$

$$\dot{i}_{r_1} = -\dot{i}_{f_1} \frac{100}{1{,}100} = -0{\cdot}091\, \dot{i}_{f_1} : v_{r_1} = v_{f_1} \frac{100}{1{,}100} = 0{\cdot}091\, v_{f_1}$$

Energy which passes on to Z_{02} is

$$v_{f_2} \dot{i}_{f_2} = 0{\cdot}91 \times 0{\cdot}091\, v_{f_1} \dot{i}_{f_1} = 0{\cdot}082\, v_{f_1} \dot{i}_{f_1}$$

that is $\underline{W_2 = 0{\cdot}082\, W_1}$

Energy absorbed by the resistance R is

$$0\cdot91^2 (i_{f_1})^2 R = 0\cdot83 \times 550 \times (i_{f_1})^2 = 455\,(i_{f_1})^2 = 455\,\frac{v_{f_1}}{Z_0}\,i_{f_1} = \underline{0\cdot91\,v_{f_1}\,i_{f_1}}$$

Reflected energy $= v_{f_1} i_{f_1} \{1 - 0\cdot082 \dot{-} 0\cdot91\} = \underline{0\cdot008\,v_{f_1}\,i_{f_1}}$

If $R = 0$: *Fig.* 17

Energy which passes on to Z_{02} is $v_{f_2} i_{f_2} = 0\cdot182\,v_{f_1} \times 1\cdot82\,i_{f_1}$;

that is $W_2 = 0\cdot33\,v_{f_1} i_{f_1} = \underline{0\cdot33\,W_1}$

so that the reflected energy is $W_1 - W_2 = \underline{0\cdot67 W_1}$

Case (*ii*) $R = 550$ ohms : *Fig.* 18

$$i_{f_2} = 0\cdot091\,i_{f_1} : v_{f_2} = 0\cdot91\,v_{f_1}$$
$$i_{r_1} = 0\cdot91\,i_{f_1} : v_{r_1} = 0\cdot91\,v_{f_1}$$

Energy which passes on to Z_{02} is

$$v_{f_2} i_{f_2} = 0\cdot091 \times 0\cdot91\,v_{f_1} i_{f_1} = \underline{0\cdot083\,v_{f_1} i_{f_1}} = \underline{0\cdot083\ W_1}$$

Energy absorbed in the resistance R is

$$R(i_{f_2})^2 = 550 i_{f_2} \frac{0\cdot91}{500} v_{f_1} = \frac{550 \times 0\cdot091 \times 0\cdot91}{500} i_{f_1} v_{f_1} = \underline{0\cdot091\,v_{f_1} i_{f_1}} = \underline{0\cdot091\ W_1}$$

Reflected energy $= v_{f_1} i_{f_1} (1 - 0\cdot083 - 0\cdot091) = \underline{0\cdot83\,v_{f_1} i_{f_1}} = \underline{0\cdot83 W_1}$

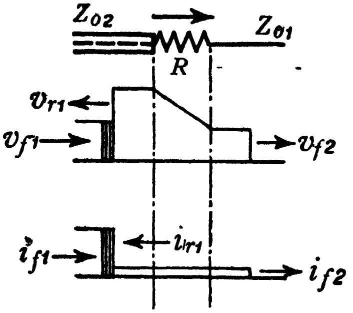


Fig. 18

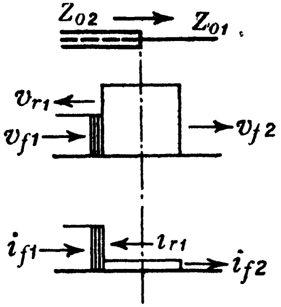


Fig. 19

If $R = 0$: *Fig.* 19

Energy which passes on to Z_{02} is

$$v_{f_2} i_{f_2} = 1\cdot82 \times 0\cdot182 \times v_{f_1} i_{f_1} = \underline{0\cdot33\,v_{f_1} i_{f_1}} = \underline{0\cdot33\ W_1}$$

Reflected energy $= v_{r_1} i_{r_1} = \underline{0\cdot67\,v_{f_1} i_{f_1}} = \underline{0\cdot67 W_1}$

11.—Show how a Four-pole Unit system may be used as a filter to block the passage of high-frequency currents.

ANSWER.—In Fig. 20, is shown a Four-pole T system comprising two equal inductances in series each of magnitude $\frac{L_T}{2}$ henry, and a capacitance of C farad connected across the lines.

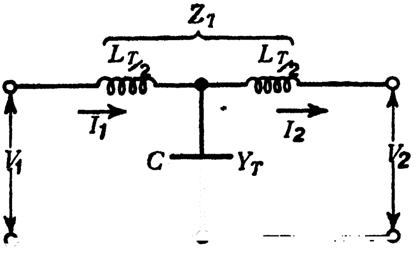


Fig. 20

The total series impedance is $\mathfrak{Z}_T = j\omega L_1$ ohms and the total shunted admittance is $\mathfrak{Y}_T = j\omega C_T$ siemens. Then it is easily shown that

$$\mathfrak{V}_1 = \frac{\mathfrak{J}_1 - \mathfrak{J}_2}{\mathfrak{Y}} + \frac{\mathfrak{J}_1\cdot\mathfrak{Z}_T}{2}$$

$$\mathfrak{V}_2 = \frac{\mathfrak{J}_1 - \mathfrak{J}_2}{\mathfrak{Y}} - \frac{\mathfrak{J}_1\cdot\mathfrak{Z}_T}{2}$$

so that,

$$\mathfrak{V}_1 = \mathfrak{V}_2\left[1 + \frac{\mathfrak{Z}_T\cdot\mathfrak{Y}_1}{2}\right] + \mathfrak{J}_2\cdot\mathfrak{Z}_T\left[1 + \frac{\mathfrak{Z}_T\cdot\mathfrak{Y}_T}{4}\right]$$

$$\mathfrak{J}_1 = \mathfrak{V}_2\cdot\mathfrak{Y}_T + \mathfrak{J}_2\left[1 + \frac{\mathfrak{Z}_T\cdot\mathfrak{Y}_T}{2}\right]$$

Writing,[*]

$$\text{Cosh}\ \mathfrak{m} \equiv 1 + \frac{\mathfrak{Z}_T\cdot\mathfrak{Y}_T}{2} : \text{ where } \mathfrak{m} \equiv b + ja$$

then,

$$\text{Sinh}^2\mathfrak{m} = \text{Cosh}^2\mathfrak{m} - 1 = \mathfrak{Y}_T\cdot\mathfrak{Z}_T\left(1 + \frac{\mathfrak{Y}_T\cdot\mathfrak{Z}_T}{4}\right)$$

and writing,

$$\mathfrak{Z}_F \equiv \sqrt{\frac{\mathfrak{Z}_T}{\mathfrak{Y}_T}}\sqrt{1 + \frac{\mathfrak{Y}_1\cdot\mathfrak{Z}_T}{4}}$$

then,

$$\left.\begin{aligned} \mathfrak{V}_1 &= \mathfrak{V}_2\ \text{Cosh}\ \mathfrak{m} + \mathfrak{J}_2\cdot\mathfrak{Z}_F\ \text{Sinh}\ \mathfrak{m} \\ \mathfrak{J}_1 &= \mathfrak{V}_2\cdot\frac{1}{\mathfrak{Z}_F}\ \text{Sinh}\ \mathfrak{m} + \mathfrak{J}_2\ \text{Cosh}\ \mathfrak{m} \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots \text{(i)}$$

These are the general equations for the 4-pole T system and they are identical in form with the general equations for a long-distance transmission line when the line resistance losses are included.

From a comparison of the form of the equations (i) with the corresponding simultaneous equations for a long-distance transmission line, it is seen that

the vector $\mathfrak{m} = b + ja$

corresponds to the " propagation constant " of the line so that,

b corresponds to the total damping factor, and

a corresponds to the total angle of phase displacement.

---

[*] The symbol $\equiv$ denotes " stands for."

The vector $\mathfrak{Z}_F$ is termed the " characteristic constant " of the filter circuit and may be written,

$$\mathfrak{Z}_F = \sqrt{\frac{L_T}{C_T}}\sqrt{1 - \omega^2\left(\frac{L_T \cdot C_T}{4}\right)}$$

and writing

$$\omega_o^2 = \frac{4}{L_T \cdot C_T}$$

then,

$$\mathfrak{Z}_F = \sqrt{\frac{L_T}{C_T}}\sqrt{1 - \frac{\omega^2}{\omega_o^2}}$$

where ω_o is the resonance frequency for the capacitance $\frac{C_T}{2}$ in series with the inductance $\frac{L_T}{2}$.

When,

$$\frac{\omega}{\omega_o} < 1 \text{ then } \mathfrak{Z}_F \text{ is real :}$$

$$\frac{\omega}{\omega_o} > 1 \text{ then } \mathfrak{Z}_F \text{ is imaginary.}$$

Now,

$$\text{Cosh } \mathfrak{m} = 1 + \frac{\mathfrak{Y}_T\mathfrak{Z}_T}{2} : \text{Sinh } \mathfrak{m} = \sqrt{\mathfrak{Y}_T\mathfrak{Z}_T}\sqrt{1 + \frac{\mathfrak{Y}_T\mathfrak{Z}_T}{4}}$$

so that,

$$\text{Sinh } \frac{\mathfrak{m}}{2} = \sqrt{\frac{\mathfrak{Y}_T\mathfrak{Z}_T}{4}} = \sqrt{-\left(\frac{\omega}{\omega_o}\right)^2} = j\,\frac{\omega}{\omega_o} \qquad \text{.................(ii)}$$

and hence,

$$\text{Sinh}\left(\frac{b}{2} + j\,\frac{a}{2}\right) = \text{Sinh } \frac{b}{2}\cos\frac{a}{2} + j \text{ Cosh } \frac{b}{2}\sin\frac{a}{2} = j\,\frac{\omega}{\omega_o}$$

Equating respectively, the real and the imaginary quantities gives,

$$\left.\begin{aligned} \text{Sinh } \frac{b}{2}\cos\frac{a}{2} &= 0 \\ \text{Cosh } \frac{b}{2}\sin\frac{a}{2} &= \frac{\omega}{\omega_o}. \end{aligned}\right\} \qquad \text{..................................(iii)}$$

The simultaneous equations (iii) have two distinct solutions which are sharply contrasted with one another as follows:

(1) The first equation of (iii) is satisfied if,

$$\operatorname{Sinh}\frac{b}{2} = 0$$

that is, $b = o$, and substituting this value of b in the second equation of (iii) gives

$$a = 2\ \sin^{-1}\frac{\omega}{\omega_o}$$

and since the sine of an angle cannot be greater than unity, this solution can only hold for $\frac{\omega}{\omega_o} =$ or <1, that is [see expression (ii)] for

$$\frac{\mathfrak{Y}_T\mathfrak{Z}_T}{4} < -1.$$

(2) The first equation of (iii) is satisfied if,

$$\frac{a}{2} = \frac{\pi}{2}$$

and substituting this value for a in the second equation of (iii) gives,

$$b = 2\ \cosh^{-1}\frac{\omega}{\omega_o}$$

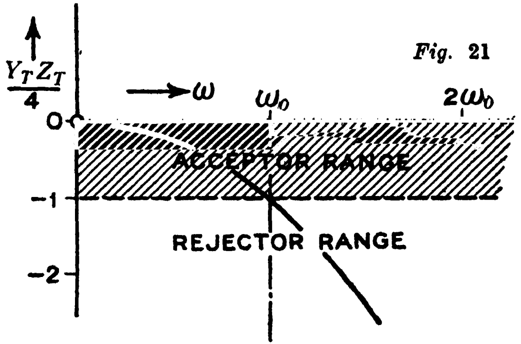


Fig. 21

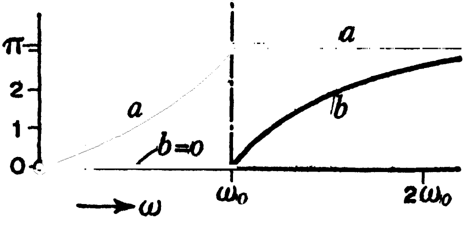


Fig. 22

Taking the following numerical data, viz.,

$$\mathfrak{Z}_T = j\omega L_T = j\omega \times 0{\cdot}02 \text{ ohm}$$

$$\mathfrak{Y}_T = j\omega C_T = j\omega \times 800 \times 10^{-6} \text{ siemens}$$

then,

$$\omega_0 = \sqrt{\frac{4}{L_T C_T}} = 500$$

$$\frac{\mathfrak{Y}_T \mathfrak{Z}_T}{4} = -\left(\frac{\omega}{\omega_0}\right)^2$$

In Fig. 21 is shown the value of $\frac{\mathfrak{Y}_T \cdot \mathfrak{Z}_T}{4}$ as a function of ω. Fig. 22 shows the respective values of the phase angle a and the damping factor b as functions of ω. It will be seen from these diagrams that for values of $\omega < \omega_0$, the damping factor is zero, whilst for $\omega > \omega_0$, the damping factor increases as a Cosh function and the phase angle remains at the constant value $a = \pi$.

The filter circuit system of Fig. 20, which is termed a "Choke Filter Circuit" is appropriate for damping high-frequency harmonics and, in particular, for purifying a D.C. supply from A.C. components.

12.—Show how a Four-pole Unit system may be used to block the passage of low-frequency currents.

ANSWER.—For this purpose, the "condenser-chain" filter circuit shown in Fig. 23 may be used and it will be seen that, for this arrangement, the following relationships will hold, viz.,

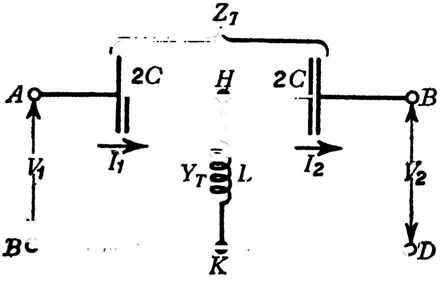


Fig. 23

$$\mathfrak{Z}_T = \frac{1}{j\omega C} = -j\frac{1}{\omega C}\,;\quad \mathfrak{Y}_T = \frac{1}{j\omega L} = -j\frac{1}{\omega L}$$

since the capacitance for the series connection of two condensers each of capacitance $2C$ is $\frac{2C \times 2C}{2C + 2C} = C$. Also,

$$\frac{\mathfrak{Z}_T \cdot \mathfrak{Y}_T}{4} = -\frac{1}{\omega^2 4L.C} = -\left(\frac{\omega_0}{\omega}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

where $\omega_0 = \sqrt{\frac{1}{4L.C}}$ and is the resonance frequency of a circuit consisting of a capacitance of $2C$ farad in series with an inductance of $2L$ henry.

The characteristic constant for this filter circuit is

$$\mathfrak{Z}_F = \sqrt{\frac{\mathfrak{Z}_T}{\mathfrak{Y}_T}}\sqrt{1+\frac{\mathfrak{Z}_T\cdot\mathfrak{Y}_T}{4}} = \sqrt{\frac{L}{C}}\sqrt{1-\left(\frac{\omega_0}{\omega}\right)^2}$$

For supply frequencies ω such that $\omega<\omega_0$, this factor $\mathfrak{Z}_F$ will be imaginary whilst for frequencies such that $\omega>\omega_0$, the factor $\mathfrak{Z}_F$ will be real.

Again, as in Example 11, writing

$$\text{Cosh}\ \mathfrak{m} = 1 + \frac{\mathfrak{Y}_{\dot{T}}\mathfrak{Z}_T}{2} : \text{Sinh}\ \frac{\mathfrak{m}}{2} = \sqrt{\frac{\mathfrak{Y}_{\dot{T}}\mathfrak{Z}_T}{4}} = j\left(\frac{\omega_0}{\omega}\right)$$

so that,

$$\text{Sinh}\left(\frac{b}{2}+j\frac{a}{2}\right) = j\frac{\omega_0}{\omega}$$

that is,

$$\text{Sinh}\ \frac{b}{2}\cos\frac{a}{2} + j\text{Cosh}\ \frac{b}{2}\sin\frac{a}{2} = j\frac{\omega_0}{\omega}$$

Equating respectively the real and the imaginary parts of this equation gives,

$$\left.\begin{aligned}\text{Sinh}\ \frac{b}{2}\cos\frac{a}{2} &= 0\\ \text{Cosh}\ \frac{b}{2}\sin\frac{a}{2} &= \frac{\omega_0}{\omega}\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

and these two relationships must be simultaneously true. As in Example 11, two separate and sharply contrasted solutions can be found for the equations (ii) as follows:

(1). *Sinh* $\frac{b}{2} = 0$, *that is* $b = 0$, will satisfy the first equation (ii), and, substituting $b = 0$ in the second equation gives,

$$a = 2\ \sin^{-1}\left(\frac{\omega_0}{\omega}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

And this solution can only apply for values of the frequency such that $\omega>\omega_0$. That is to say, when $\omega>\omega$, the damping b is zero. From expression (i) the conditions for this solution imply that $\frac{\mathfrak{Y}_T\cdot\mathfrak{Z}_T}{4} < -1$.

(2) *The first equation of (ii) will also be satisfied when cos* $\frac{a}{2} = 0$, *that is when* $a = \pi$, and substituting this value for a in the second equation gives

$$b = 2 \text{ Cosh}^{-1}\left(\frac{\omega_0}{\omega}\right) \quad \ldots\ldots\ldots\ldots \quad \text{(iv)}$$

and since this expression for b shows that $\frac{\omega_0}{\omega}$ cannot be less than 1, this solution (iv) can only apply for frequencies such that $\omega < \omega_0$.

Applying the following numerical data to these respective solutions, viz.,

$$2C = \frac{2 \times 50}{10^6} \text{ farad} : L = 0.02 \text{ henry}, \ \omega_0 = 500$$

the diagrams shown in Figs. 24 and 25 have been obtained. It will be seen from these diagrams that currents of frequency less than $\omega = \omega_0 = 500$ are heavily damped, whilst for currents of frequency greater than this critical value, the damping is zero.

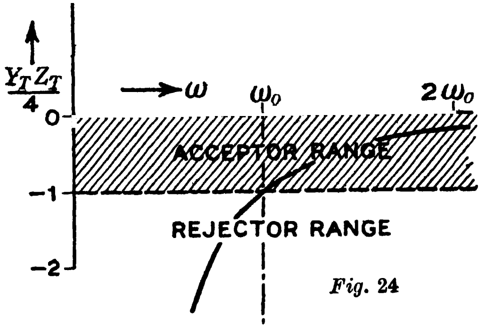


Fig. 24

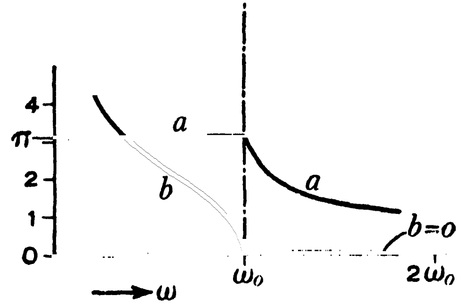


Fig. 25

13.—*Show how a Four-pole T system may be used as a filter to pass a selected band of frequencies.*

Answer.—The arrangement of the T circuit system for this type of filter is shown in Fig. 26, from which it will be seen that the following relationships will hold, viz.,

$$\mathfrak{Z}_T = j\omega L - j\,\frac{1}{\omega\cdot K}$$

$$\mathfrak{Y}_T = j\omega\cdot C - j\,\frac{1}{\omega\cdot M}$$

so that,

$$\mathfrak{Y}_T\mathfrak{Z}_T = -\tfrac{1}{4}\left(\omega\cdot L - \frac{1}{\omega\cdot K}\right)\left(\omega\cdot C - \frac{1}{\omega\cdot M}\right)$$

that is,

$$\mathfrak{Y}_T\mathfrak{Z}_T = -\frac{1}{\omega_o^2}\left(\omega - \frac{1}{\omega\cdot L\cdot K}\right)\left(\omega - \frac{1}{\omega\cdot M\cdot C}\right)$$

where

$$\omega_o^2 = \frac{4}{L\cdot C}$$

If the respective values of L, K, M, and C are now chosen so that*

$$L\cdot K = M\cdot C \equiv \frac{1}{\Omega^2} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

then

$$\frac{\mathfrak{Y}_T\mathfrak{Z}_T}{4} = -\frac{1}{\omega_o^2}\left(\omega - \frac{\Omega^2}{\omega}\right)^2 \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

Now it has been seen in Examples 11 and 12 that the frequencies which the filter will be able to pass will be such that the corresponding values of the quantity $\frac{\mathfrak{Y}_T Z_T}{4}$ must lie within the range of values from 0 to -1. These limiting values determine the range of frequencies which the filter will pass, viz.,

$$\frac{\mathfrak{Y}_T\mathfrak{Z}_T}{4} = -1 = -\frac{1}{\omega_o^2}\left(\omega - \frac{\Omega^2}{\omega}\right)^2$$

that is,

$$\omega^2 \mp \omega.\omega_o - \Omega^2 = 0 \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

---

*The symbol $\equiv$ denotes "is represented by"

The values of the limiting frequencies of the acceptor range will therefore be given by the solution of this quadratic equation, viz.,

$$\left.\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right\} = \pm\frac{\omega_o}{2} \pm \sqrt{\frac{\omega_o^2}{4} + \Omega^2}$$

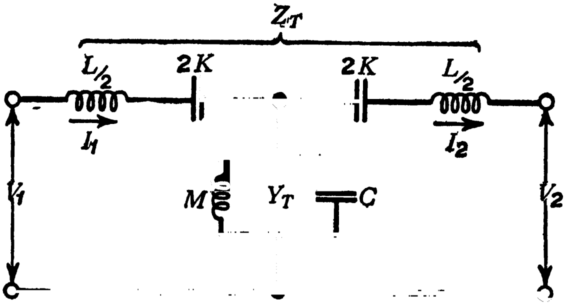


Fig. 26

and since only positive values of the frequency have any practical significance, the solutions may be written

$$\left.\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right\} = \sqrt{\frac{\omega_o^2}{4} + \Omega^2} \mp \frac{\omega_o}{2} \quad \text{(iv)}$$

That is to say, the range of frequencies which the filter will pass is

$$\omega_2 - \omega_1 = \omega_o = \frac{2}{\sqrt{L \cdot C}}$$

From expression (ii) it will be seen that

$$\frac{\mathfrak{Y}_T . \mathfrak{Z}_T}{4} = 0, \text{ when } \omega = \frac{\Omega^2}{\omega}$$

that is, when

$$\omega = \Omega = \frac{1}{\sqrt{L \cdot K}} = \frac{1}{\sqrt{M \cdot C}} \quad \text{(v)}$$

Thus, if

$L = 0.02$ henry : $C = 800 \times 10^{-6} F$: $K = 50 \times 10^{-6} F$: $M = 0.125 \times 10^{-2}$ henry, then,

$$\omega_o = \frac{2}{\sqrt{L \cdot C}} = 500.$$

$$\Omega = \frac{1}{\sqrt{L \cdot K}} = \frac{1}{\sqrt{M \cdot C}} = 1{,}000.$$

The limiting values for the acceptor range of frequencies will therefore be:

$$\omega_2 = \sqrt{\frac{\omega_o^2}{4} + \Omega^2} + \frac{\omega_o}{2} = 1{,}280$$

$$\omega_1 = \sqrt{\frac{\omega_o^2}{4} + \Omega^2} - \frac{\omega^{\circ}}{2} = 780$$

The damping factor b and the phase displacement angle a may be found by the same method as in Examples 11 and 12; thus,

$$\text{Cosh}\ \mathfrak{m} = 1 + \frac{\mathfrak{Y}_T \cdot \mathfrak{Z}_1}{2}$$

$$\text{Sinh}\ \frac{\mathfrak{m}}{2} = \text{Sinh}\left(\frac{b}{2} + j\frac{a}{2}\right) = \sqrt{\frac{\mathfrak{Y}_1 \cdot \mathfrak{Z}_T}{4}} = j\frac{1}{\omega_o}\left(\omega - \frac{\Omega^2}{\omega}\right)$$

so that,

$$\text{Sinh}\left(\frac{b}{2} + j\frac{a}{2}\right) = \sinh\frac{b}{2}\cos\frac{a}{2} + j\ \text{Cosh}\ \frac{b}{2}\sin\frac{a}{2} = j\frac{1}{\omega_o}\left(\omega - \frac{\Omega^2}{\omega}\right)$$

By equating respectively, the real and the imaginary quantities in this equation it is seen that the following two equations must simultaneously be satisfied, viz.,

$$\left.\begin{aligned} &\text{Sinh}\ \frac{b}{2}\cos\frac{a}{2} = 0 \\ &\text{Cosh}\ \frac{b}{2}\sin\frac{a}{2} = \frac{1}{\omega_o}\left(\omega - \frac{\Omega^2}{\omega}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \text{(vi)}$$

Two separate solutions of these equations are then obtained as follows:

(1) *Sinh* $\frac{b}{2} = 0$, *that is*, $b = 0$. This satisfies the first equation of (vi), and substituting in the second equation it is found that

$$a = 2\sin^{-1}\left[\frac{1}{\omega_o}\left(\omega - \frac{\Omega^2}{\omega}\right)\right]$$

and clearly, this solution can only hold for a range of frequencies such that

$$\frac{1}{\omega_o}\left(\omega - \frac{\Omega^2}{\omega}\right) < 1.$$

From the foregoing investigation [see expression (ii)] it will be seen that this range of frequencies is covered by the acceptor range of frequencies, that is, the range for which the value of $\frac{\mathfrak{Y}_T \cdot \mathfrak{Z}_T}{4} < -1$.

(2) *The first equation of (vi) is satisfied if* $\cos\frac{a}{2} = 0$, *that is, if* $a = \pi$.

If this value for a is substituted in the second equation of (vi) it is found that,

$$b = 2\,\mathrm{Cosh}^{-1}\left[\frac{1}{\omega_0}\left(\omega - \frac{\Omega^2}{\omega}\right)\right]$$

This solution requires that the quantity $\left[\frac{1}{\omega_0}\left(\omega - \frac{\Omega^2}{\omega}\right)\right] > 1$ and

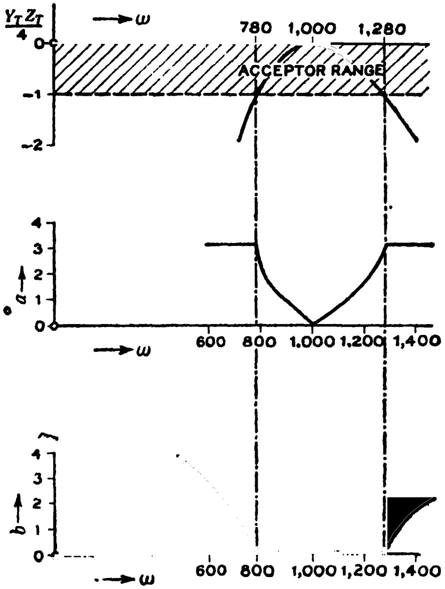


Fig. 27

consequently this solution can only apply to those values of the frequency ω which are outside the acceptor range.

Making use of the numerical data given in the foregoing for the circuit constants, Fig. 27 shows the values of $\frac{\mathfrak{Y}_T \cdot \mathfrak{Z}_T}{4}$ as a function of ω, and the manner in which the damping b and the phase angle a depend upon the frequency ω is also shown in Fig. 27.

14.—A single-phase overhead line of bronze conductors 3 mm. diameter is used for high-frequency transmission. Calculate the damping factor as a function of the frequency.

ANSWER.—The specific resistance for bronze conductors may be taken to be 2.9×10^{-6} ohm per cm. cube or, expressed in electromagnetic units,

$$\rho_{e.m} = 2.9 \times 10^{3} \text{ per c. cube in e.m. units.}$$

The resistance to d.c. per km. run of the transmission line is

$$R_{d\,c} = \frac{2.9}{10^6} \times \frac{2 \times 10^5}{\pi \times 0{\cdot}15^2} = 2 \times 4{\cdot}1 = \underline{8{\cdot}2 \text{ ohms}} \text{ per km. run.}$$

Again, making use of expression (64) Chapter XIII, page 429 "Principles of Electrical Engineering," then for a frequency f = 250 k.-hz,

$$y = \pi a \sqrt{\frac{f.\mu}{\rho_{e\,m}}} = \pi \times 0.15 \sqrt{\frac{25 \times 10^4}{2.9 \times 10^3}} = 4{\cdot}35$$

Hence, the resistance to a.c. is

$$R_{a\,c.} = R_{d.c}\,(y + \tfrac{1}{4}) = 4{\cdot}6\, R_{d\,c}$$

that is

$$R_{a\,c} = 4{\cdot}6 \times 8{\cdot}2 = 37{\cdot}6 \text{ ohms per km. run of line.}$$

The damping factor is then

$$\beta = \frac{R_{a\,c}}{2}\sqrt{\frac{C}{L}} = \frac{R_{a.c.}}{2Z_0}$$

and taking $Z_0 = 600$ ohms

$$\beta = \frac{37.6}{1{,}200} = 0.031 \text{ neper per km. run of line.}$$

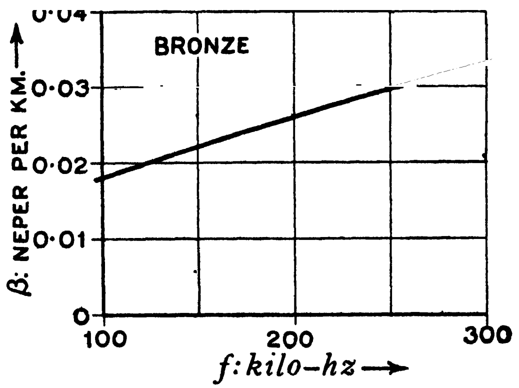


Fig. 28

In Fig. 28 are shown the test values for the damping factor as a function of the frequency. It is to be observed that the effects of weather, e.g., snow and ice on the line, and leakage of the insulators will produce additional damping ; that is, the influence of the component

$$\frac{G}{2}\sqrt{\frac{L}{C}}$$

of the expression (iv), Question (18) cannot be neglected.

15.—How would you measure the damping factor of a high-frequency cable?

Answer.—In Fig. 29 *a b* and *c d* represent a pair of strands of a telephone cable the length of the pair being *s* km. as shown in the diagram. If Z_0 ohms is the surge impedance, a non-inductive resistance $R_0 = Z_0$ is connected, across the terminals *b d*. The ends *a c* are connected to a source of variable frequency alternating current so that a.c. power will be transmitted to the load resistance R_0, and since this resistance is equal to the surge impedance, the load will be the " natural load " of the cable strands. Consequently there will be no reflection effects at the terminals *b d* and only forward travelling waves of pressure and current will then appear in the lines.

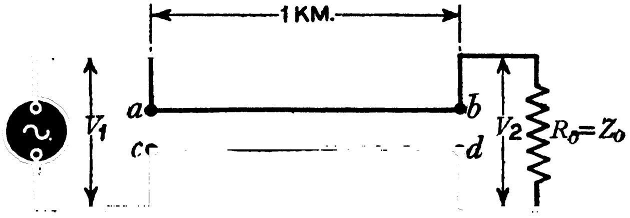


Fig. 29

The pressure V_2 at the consumer's end will then be related to the pressure V_1 at the sending end, by the equation

$$V_1 = V_2\left(e^{\beta 2s}\right)$$

where β neper per km. is the damping factor and $2s$ km. is the total length of the pair of strands for a transmission distance of *s* km. Hence

$$\log_e\left(\frac{V_1}{V_2}\right) = \beta.2s$$

and

$$\beta = \frac{1}{2.s}\log_e\left(\frac{V_1}{V_2}\right) \text{ neper per km.}$$

If therefore the ratio of the pressures at the sending end and the receiving end of the pair of strands is known when the natural load is being transmitted, the damping factor β can be found at once.

It was formerly believed that the transmission of high-frequency currents by means of normal telephone cables was impracticable on account of the heavy damping which would be produced. More detailed examination of the possibilities of telephone cables from this point of view, however, showed that since the " loss angles " (see Chapters II and XI, pages 68 and 362 of " Principles of Electrical Engineering ")

$$\tan \varepsilon = \frac{R}{\omega L} : \tan x = \frac{G}{\omega C}$$

became respectively small at high frequencies since the numerators of these fractions increase much less rapidly than the denominators and the damping factor can be expressed in the approximate form

$$\beta = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}} \text{ neper per km. } \dots\dots\dots\dots\dots \quad (1)$$

where $\sqrt{\frac{L}{C}}$ is the surge impedance Z_0. The resistance R ohm per km. depends upon the skin effect, and the capacitance will vary with the frequency in consequence of the dielectric loss.

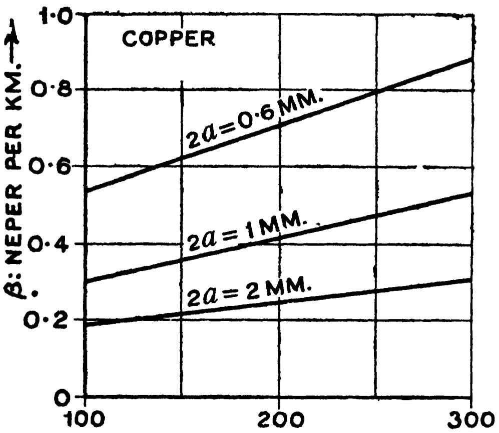


Fig. 30

Actual measurements of the damping factor for such a pair of telephone cable strands shows that the foregoing expression gives the value of β to a good degree of approximation. Thus, in Fig. 30 is shown the value of β as a function of the frequency for a series of strand conductors of respective diameters $2a = 0{\cdot}6$ mm., 1·0 mm., and 2 mm. respectively. The surge impedance is $Z_0 = 150$ ohms in each case and the numerical calculation for β is indicated by the following example.

For copper strands each 1 km. long and 0·6 mm. diameter the total length of the circuit is 2 km. and the resistance of the strands to d.c. is

$$R_{dc} = \rho \frac{l}{q} \text{ ohms}$$

where $\rho = 1{\cdot}83 \times 10^{-6}$ ohm per cm. cube : $l = 2 \times 10^5$ cm.

$$q = \frac{\pi}{4} \times (0{\cdot}06)^2 = 0{\cdot}00283 \text{ sq. cm.}$$

Hence $$R_{dc} = \frac{1{\cdot}83}{10^6} \times \frac{2 \times 10^5}{0{\cdot}00283} = 129 \text{ ohms}$$

The resistance to a.c. may be found by means of the formulæ given in

Chapters XIII and XIV of " Principles." For example, making use of the expressions (64) and (65), Chapter XIII, page 429 :

$$y = \pi a \sqrt{\frac{f.\mu}{\rho_{e\,m.}}}$$

where $\rho_{e.m.} = 1{\cdot}83 \times 10^3$ is the specific resistance per c. cube in electromagnetic units, so that, for $f = 250{,}000$ hz.,

$$y = \pi \times 0{\cdot}03 \sqrt{\frac{250 \times 10^3}{1{\cdot}83 \times 10^3}} = 0{\cdot}094 \quad \sqrt{136} = 1{\cdot}10$$

Hence the resistance to a.c. is

$$R_{a.c.} = R_{d.c.}\,[y + \tfrac{1}{4}] = 129 \times 1{\cdot}35 = 175 \text{ ohms}$$

for a total length of the cable strands equal to 2 km. Neglecting the damping effect due to the conductance G in the expression (1) then

$$\beta = \frac{R_{a.c.}}{2} \times \frac{1}{Z_o} = \frac{175}{2 \times 150} = 0{\cdot}58 \text{ neper per km. run of the cable.}$$

From the actual test data given in Fig. 30 it will be seen that the measured value for β in this case is 0·8 neper per km. run of the cable, and since the damping effect of the conductance G has been neglected in the foregoing numerical evaluation, the agreement is sufficiently good to show the value of this method for calculating β.

16.—Discuss the significance of the damping factor as regards the question of the installation of amplifiers for the operation of a high-frequency transmission line.

ANSWER.—If it be assumed that a damping factor of 6 nepers is permissible for an unamplified section of the line, then the results given in Fig. 30 for the solution of Example 15 show that from 7 to 20km. run of cable can be operated without amplification, whilst the results given in Fig. 28 for the solution of Example 14 show that from 20 to 35 km. run of an overhead line can be operated without amplification.

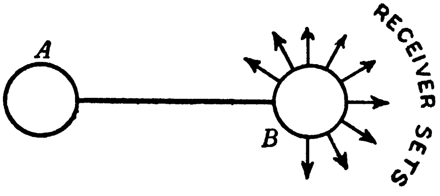


Fig. 31

The question as to how and when amplifiers should be introduced into a high-frequency transmission line can be answered by means of the following considerations : Suppose in Fig. 31 B represents a receiving station distant s=5 km. from a sending station A and assume that 100 receiver sets are connected to B, the transmission line which connects A and B

having a damping factor of $\beta=0.8$ neper per km. If the pressure V_2 at the receiver station B is to be not less than 0.1 volt, then the requisite pressure V_1 at the sending station A will be

$V_1 = V_2 e^{\beta s} = 0{\cdot}1\, e^{0.8\times5} = 0.1 \times e^4 = 5.45$ volts .... (i)

and if the surge impedance of the line is $Z_0 = 150$ ohms (the line being matched so that the power transmitted is its natural power), then the current which must be supplied at the sending end A when only one receiver set is connected at B is

$I_1 = \frac{V_1}{Z_0} = \frac{5.45}{150} = 0.0364$ ampere .... (ii)

and, consequently, the power required at the sending end of the line to supply one receiver set is

$W_1 = V_1.I_1 = 5.45\times0.0364 = 0.2$ watt .... (iii)

and the power taken by the receiver set is

$W_2 = V_2.I_2 = 0.1 \times I_1 e^{-\beta s} = 0.1 \times 0.0364\times e^{-4}$

that is

$W_2 = 0.1 \times 0.0364 \times 0.0183 = 6{\cdot}7 \times 10^{-5}$ watts ....(iv)

alternatively expressed,

$W_2 = W_1\, e^{-2\beta s} = W_1\, e^{-8} = 0.2 \times e^{-8} = 6.7 \times 10^{-5}$ watts

The foregoing calculations show that in order to supply one receiver set at B the power which must be supplied at A is 0.2 watts and consequently for 100 receiver sets at B the power which must be supplied at A is

$W_1 = 20$ watts .... (v)

Suppose now that an amplifier is installed at B in such a position that the mean distance of the amplifier from the 100 receiver sets is 1 km. Then, since the pressure at the terminals of the receiver sets must still be 0.1 volt, the pressure at the output terminal of the amplifier must be

$V_{\text{amp}} = 0.10 \times e^{\beta s} = 0.10 \times e^{0.8\times1} = 0.10 \times e^{0.8} = 0.22$ volt,

and if the line connecting the receiver sets to the amplifier has a surge impedance $Z_0 = 150$ ohms the power required from the amplifier for one receiver set is

$$W_{\text{amp}} = \frac{(V_{\text{amp}})^2}{150} = \frac{0.22^2}{150} = 0.00032 \text{ watt}$$

and for 100 receiver sets the amplifier will be required to supply

0.032 watt.

To operate the amplifier the station A will require a supply of 0.2 watt and consequently the total power now required for the 100 receiver sets will be

$0.032+0.2 = 0.232$ watt .... (vi) and this is to be compared with the 20 watts (expression (v)) which must be supplied when no amplifier is used.

The foregoing results show that it is advantageous to de-centralise the generation of the high-frequency power as much as possible, that is to say, to sectionalise the generation into a number of stages.

17.—For the transmission cable such as that shown in Fig 29 Example 15, find what power is dissipated per metre length of the cable when the consumer is receiving 1 watt.

ANSWER.—If the cable is matched so that the power supplied to the consumer is the natural power of the cable, then at any distance s from the receiver's terminals the power carried by the cable is

$$W = \frac{V_2^2}{Z_0} e^{2\beta s} \text{ watts} \qquad \text{(i)}$$

where V_2 is the pressure at the consumer's end Z is the surge impedance of the cable, and β the damping factor per unit length.

If, for example, $\beta = 0.0015$ neper per metre, then,

$$W = \frac{V_2^2}{Z_0} e^{0\cdot003S} \text{ watts}$$

where s metres is the distance from the consumer's end.

Differentiating the expression (i) with respect to s gives

$$\frac{\triangle W_S}{\triangle s} = 2\beta \cdot W_s$$

that is,

$$\triangle W_s = 2\beta \cdot W_s. \times \triangle s \qquad \text{(ii)}$$

At the receiver's end of the line,

$$W_s = W_2 = 1 \text{ watt}$$

and

$$s = 0$$

so that substituting the following numerical data in (ii) viz.,

$W_s = 1$ watt ; $2\beta = 0\cdot0023$; $\triangle s = 1$ metre

then

$\triangle W_s = 0\cdot0023 \times 1 \times 1 = 0\cdot0023$ watt per metre.

At a place in the line which is 3*km. from the consumer's end,*

$$W_s = W_2 e^{2\beta s}$$

where $2\beta s = 0\cdot0023 \times 3{,}000 = 6\cdot9$

so that
$$W_s = 1 \times e^{6 \cdot 9} = 1{,}000 \text{ watts}$$

and the power dissipated at this place in the cable is [see expression (ii)],
$$\triangle W = 0 \cdot 0023 \times 1000 \times 1$$

that is
$$\triangle W_s = 2 \cdot 3 \text{ watts per metre}$$

Whether a cable can be allowed to dissipate power at such a rate is a question which must be decided by the manufacturers of the particular cable used.

18.—*Show how: (1) the damping of a cable may be measured, and (2) how the damping of a cable may be derived from the Q factor and the dielectric loss factor of the cable, viz., tan δ.*

ANSWER. The operational damping b of a cable is determined by the ratio of the power W_1 at the sending end to the power W_2 at the receiving end when the cable is supplying its natural power. That is,

$$\frac{W_1}{W_2} = \frac{(V_2 e^{\beta l})^2}{Z_0 \cdot W_2} = e^{2b}$$

so that
$$b = \tfrac{1}{2} \log_e \frac{W_1}{W_2} \quad \text{(i)}$$

Since the cable is assumed to be carrying its natural power,
$$W_1 = V_1 . I_1 : \ W_2 = V_2 . I_2$$

and consequently,
$$b = \tfrac{1}{2} \log_e \left(\frac{W_1}{W_2}\right) = \tfrac{1}{2} \log_e \left(\frac{V_1}{V_2}\right)^2 = \tfrac{1}{2} \log_e \left(\frac{I_1}{I_2}\right)^2$$

that is,
$$b = \log_e \frac{V_1}{V_2} = \log_e \frac{I_1}{I_2} \quad \text{(ii)}$$

Thus, if
$$\frac{W_1}{W_2} = 20$$

$$b = \tfrac{1}{2} \log_e 20 = \tfrac{1}{2} \times 2 \cdot 3 \times \log_{10} 20 = \tfrac{1}{2} \times 2 \cdot 3 \times 1 \cdot 3 = \underline{1 \cdot 5 \text{ nepers}}$$

or
$$b = \log_{10} 20 = \underline{1 \cdot 3 \text{ bels}}$$

that is
$$1 \text{ bel} = 1 \cdot 15 \text{ nepers} = 10 \text{ decibels} \quad \text{(iii)}$$

(2) The damping factor β of a cable is given very approximately by the expression

$$\beta = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}} \quad \text{..........................} \quad \text{(iv)}$$

or

$$\beta = \frac{R}{2Z_o} + \frac{GZ_o}{2}$$

where $Z_o = \sqrt{\frac{L}{C}}$ and is the surge impedance.

The conductance G is given by the expression

$$G = \omega C \tan \delta$$

where δ is the " angle of loss " of the cable dielectric.

Writing expression (iv) in the form

$$\beta = \frac{R}{2}\frac{\sqrt{L.C}}{L} + \frac{G}{2}\frac{\sqrt{L.C}}{C} \quad \text{..............................} \text{(v)}$$

and noting that

$$f.\lambda = c = \frac{1}{\sqrt{L.C}} : \; \omega = 2\pi f$$

where c is the velocity of travel of electricity in the cable,

so that

$$\sqrt{L.C} = \frac{1}{f.\lambda} = \frac{2\pi}{\omega.\lambda}$$

the expression (v) may now be written

$$\beta = \frac{R.\pi}{\omega.L.\lambda} + \frac{G.\pi}{\omega.C.\lambda}$$

or

$$\beta = \frac{\pi}{\lambda}\left[\frac{R}{\omega.L} + \frac{G}{\omega.C}\right]$$

that is,

$$\beta = \frac{\pi}{\lambda}\left[\frac{1}{Q} + \tan \delta\right] \quad \text{...........................} \text{(vi)}$$

19.—What is meant by the "attenuation" of a transmission line? Show by means of a numerical example the effect of the attenuation on the power transmitted.

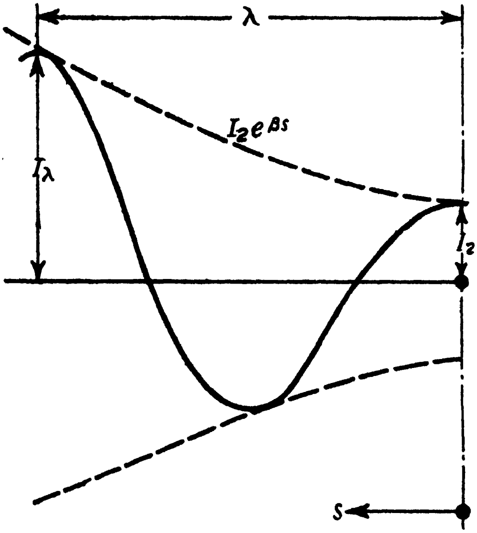


Fig. 32

Answer.—When a line is transmitting its natural power, there will be no reflection of the current wave or of the pressure wave at the consumer's terminals. That is to say, there will only be a forward travelling wave of current and a forward travelling wave of pressure: each of these waves being damped as shown in Fig. 32 for the current wave, where β is the damping factor in nepers per km.

If the current at the consumer's end of the line is I_2 r.m.s. amperes, then the current in the line at a point distant one wavelength λ km. from the consumer's end will be

$$I_\lambda = I_2 e^{\beta\lambda}$$

The *attenuation* of the line per wavelength is then,

$$\log_e \frac{I_\lambda}{I_2} = \log_e e^{\beta\lambda} = \beta.\lambda \text{ neper}$$

That is to say, the attenuation per km. is

$$\beta \text{ nepers} = 8{\cdot}7\ \beta \text{ decibels} = 0{\cdot}87\ \beta \text{ bels}$$

and is the damping factor in neper/km.

If $W_2 : V_2 : I_2$ define respectively the power, the r.m.s. pressure, and the r.m.s. current at the consumer's terminals, then, at a distance s km. from that end of the line, the current will be $I_2{\cdot}e^{\beta s}$ and the pressure will be $V_2{\cdot}e^{\beta s}$. The power at any distance s from the consumer's end will be

$$W_s = V_2{\cdot}\ e^{\beta s}{\cdot}I_2{\cdot}e^{\beta s} \text{ watts}$$

and since, for natural power transmission,

$$V_2 = I_2{\cdot}Z_0$$

where Z_0 is the surge impedance of the line, the power will be given by the expression

$$W_s = \frac{V_2^2}{Z_0} e^{2\beta s} = W_2\ e^{2\beta s} \text{ watts.}$$

A paper insulated twin telegraph line of copper wire 1mm. diameter

.and carrying a current of 1 Mhz. (10^6hz) frequency will have a damping factor of about $\beta = 1{\cdot}15$ nepers per km. so that, for :

$$s = 3 \text{ km.}: W_s = W_2\, e^{2\cdot 3 \times 3} = W_2\, e^{6\cdot 9} = 1000\, W_2$$
$$s = 6 \text{ km.}: W_s = W_2\, (e^{6\cdot 9})^2 = 10^6\, W_2$$
$$s = 9 \text{ km.}\cdot\ W_s = W_2\, (e^{6\cdot 9})^3 = 10^9\, W_2$$

That is to say, for a line 3 km. long the receiver obtains only 1/1000th part of the power which is supplied to the sending end of the line ; for a line 6 km. long the receiver only obtains one-millionth part of the power which is supplied to the sending end.

20.- ***symmetrical 4-pole T system of non-inductive resistances is shown in the accompanying diagram. Obtain the two simultaneous equations which relate the input pressure and current respectively, with the output pressure and current.***

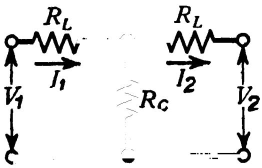


Fig. 33

ANSWER.—From Fig. 33 the following relationships are easily obtained, viz.,

$$\left.\begin{aligned} V_1 - V_2 &= I_1 \cdot R_L + I_2 \cdot R_L \\ V_1 - I_1 \cdot R_L &= (I_1 - I_2)\mathrm{R}_C \end{aligned}\right\}$$

Hence.

$$V_1 - V_2 = I_2 \cdot R_L + R_L \left\{\frac{V_1 + I_2 \cdot Rc}{Rc + R_L}\right\}$$

$$V_1 = V_2 + I_2 \cdot R_L + V_1\ \frac{R_L}{Rc + R_L}\ + I_2\ \frac{Rc \cdot R_L}{Rc + R_L}$$

$$V_1 \left(1 - \frac{R_L}{Rc + R_L}\right) = V_2 + I_2 \left(R_L + \frac{Rc \cdot R_L}{Rc + R_L}\right)$$

that is.

$$V_1\, \frac{Rc}{Rc + R_L} = \mathrm{V}_2 + \mathrm{I}_2 \left(\frac{2Rc \cdot R_L + R_L{}^2}{Rc + R_l}\right)$$

so that

$$\underline{V_1 = V_2 \left(\frac{Rc + R_L}{Rc}\right) + I_2 \cdot \frac{R_L}{Rc}\left(2\mathrm{R}c + R_L\right)} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

Again,

$$I_1 \cdot R_L = V_1 - V_2 - I_2 \cdot R_L$$

$$= V_2\left(\frac{Rc+R_L}{Rc}\right) + I_2\frac{R_L}{Rc}\left(2Rc+R_L\right) - V_2 - I_2\cdot R_L$$

that is,

$$I_1\cdot R_L = V_2\left(\frac{Rc+R_L}{Rc} - 1\right) + I_2\left(2R_L + \frac{R^2_L}{Rc} - R_L\right)$$

$$= V_2\cdot\frac{R_L}{Rc} + I_2\cdot R_L\left(1+\frac{R_L}{Rc}\right)$$

so that

$$\underline{I_1 = V_2\cdot\frac{1}{Rc} + I_2\left(1+\frac{R_L}{Rc}\right)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The required pair of simultaneous equations is therefore

$$\left.\begin{aligned} V_1 &= V_2\left(\frac{Rc+R_L}{Rc}\right) + I_2\cdot\frac{R_L}{Rc}\,(2Rc+R_L) \\ I_1 &= V_2\cdot\frac{1}{Rc} + I_2\left(1+\frac{R_L}{Rc}\right) \end{aligned}\right\} \begin{aligned} &\ldots\ldots\ldots\ldots\ldots (\text{i}) \\ &\ldots\ldots\ldots\ldots\ldots (\text{ii}) \end{aligned}$$

It is of interest to compare these equations with the corresponding pair for a no-loss a.c. transmission line, viz.,

$$\left.\begin{aligned} \mathfrak{V}_1 &= \mathfrak{V}_2 \cos\alpha s + \mathfrak{I}_2\, \mathrm{j}\, Z_o \sin\alpha s \\ \mathfrak{I}_1 &= \mathfrak{V}_2 \frac{1}{Z_o}\, \mathrm{j} \sin\alpha s + \mathfrak{I}_2 \cos\alpha s \end{aligned}\right\} \begin{aligned} &\ldots\ldots\ldots\ldots\ldots\ldots (a) \\ &\ldots\ldots\ldots\ldots\ldots\ldots (b) \end{aligned}$$

It will be seen that if the coefficient of $\mathfrak{I}_2$ in equation (*a*) is divided by the coefficient of $\mathfrak{V}_2$ in equation (*b*) the result is Z_o^2, where Z_o is the surge impedance of the line, viz.,

$$Z_o = \sqrt{\frac{L}{C}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{iii})$$

Similarly, if the coefficient of I_2 in equation (i) is divided by the coefficient of V_2 in equation (ii) the result is $R_L(2Rc+R_L)$ and the square-root of this quantity is termed the " characteristic resistance R_o," that is

$$\underline{R_o = \sqrt{R_L(2Rc+R_L)}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{iv})$$

If, then, the resistance R_o is connected to the output terminals of the *T* system it follows, from the corresponding results for the no-loss a.c. transmission line, that

$$\frac{V_1}{I_1} = \frac{V_2}{I_2} = R_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$

and,

$$\frac{V_2}{V_1} = \frac{I_2}{I_1} = k, \text{ i.e., the `` attenuation constant ''}$$

where

$$k = \frac{R_0 \cdot Rc}{R_L \cdot Rc + (R_L + Rc)(R_L + R_0)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(vi)}$$

21.—*A symmetrical 4-pole π system of non-inductive resistances is shown in the accompanying diagram. Obtain the two simultaneous equations relating the input pressure and current respectively with the output pressure and current.*

ANSWER.—The following relationships are easily obtained from the diagram of Fig. 34:

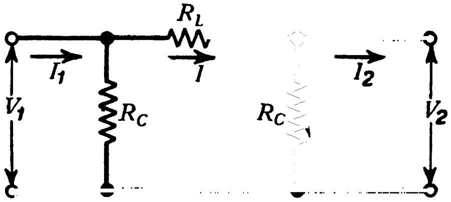


Fig. 34

$$V_1 - V_2 = I \cdot R_L :$$

$$V_1 = (I_1 - I)\, Rc : \quad V_2 = (I - I_2)\, Rc$$

$$I = \frac{V_2}{Rc} + I_2 = \frac{V_2 + I_2 \cdot Rc}{Rc}$$

$$V_1 + V_2 = (I_1 - I_2)\,\text{Rc} : \quad V_1 - V_2 = I \cdot R_L = \left[\frac{V_2 + I_2 \cdot Rc}{Rc}\right] R_L :$$

so that

$$V_1 = V_2\left(1 + \frac{R_L}{Rc}\right) + I_2 \cdot R_L \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Also,

$$I_1 = \frac{V_2 + V_1}{Rc} + I_2 = \frac{V_2}{Rc} + I_2 + V_2\left(1 + \frac{R_L}{Rc}\right) + I_2 \cdot \frac{R_L}{Rc}$$

that is,

$$\underline{I_1 = V_2\left(\frac{R_L + 2Rc}{Rc^2}\right) + I_2\left(\frac{R_L + Rc}{Rc}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

The equations (i) and (ii) are then the two simultaneous equations required, and the "characteristic resistance," as already defined in the solution of Example 20, is

$$R_o = \sqrt{\frac{R_L \cdot Rc^2}{R_L + 2Rc}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

If the resistance R_o is connected to the output terminals of the π system of Fig. 34 then,

$$\frac{V_1}{I_1} = \frac{V_2}{I_2} = R_o$$

and the "attenuation constant" is

$$k = \frac{V_2}{V_1} = \frac{I_2}{I_1}$$

where

$$\underline{k = \frac{R_o \cdot Rc}{R_o \cdot Rc + R_o \cdot R_L + Rc \cdot R_L}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

If W_1 is the input power and W_2 is the output power, then W_1 and W_2 are said to differ by N bels when,

$$\frac{W_1}{W_2} = 10^N$$

that is.

$$\log_{10} \frac{W_1}{W_2} = N \text{ bels}$$

Two powers P_1 and P_2 are said to differ by n decibels when.

$$\frac{P_1}{P_2} = 10^{\frac{n}{10}}$$

so that

$$n = 10 \log_{10} \frac{P_1}{P_2} \text{ decibels}$$

In practice, the input power W_1 and the output power W_2 will frequently be compared by measuring the pressures developed across a given impedance (or given resistance for a non-inductive system) or by measuring the currents through a given impedance.

If the input impedance and the output impedance of a 4-pole system are equal, that is, if

$$\frac{\mathfrak{B}_1}{\mathfrak{J}_1} = \frac{\mathfrak{B}_2}{\mathfrak{J}_2} = Z_o \text{ (or } R_o) \quad \ldots\ldots\ldots\ldots \quad \text{(v)}$$

then,

$$\frac{W_1}{W_2} = \left(\frac{V_1}{V_2}\right)^2 = \left(\frac{I_1}{I_2}\right)^2$$

and the " attenuation constant " is

$$k = \frac{V_2}{V_1}$$

Where k has the value given in expression (iv), then

$$\text{the ‘ attenuation ''} = \log_{10}\left(\frac{W_1}{W_2}\right) \text{ bels} = \log_{10}\left(\frac{V_1}{V_2}\right)^2 \text{ bels}$$

$$= 2 \log_{10}\left(\frac{V_1}{V_2}\right) \text{ bels}$$

or,

$$\text{the `` attenuation ''} = 2 \log_{10}\left(\frac{1}{k}\right) \text{ bels}$$

Alternatively,

$$\text{the `` attenuation ''} = \tfrac{1}{2} \log_e\left(\frac{W_1}{W_2}\right) \text{ nepers}$$

$$= \log_e\left(\frac{V_1}{V_2}\right) \text{ nepers}$$

$$= 2{\cdot}3 \log_{10}\left(\frac{V_1}{V_2}\right) \text{ nepers}$$

so that

1 bel= 1.15 nepers.

Chapter XVI

THE PROPAGATION OF ELECTROMAGNETIC WAVES THROUGH SPACE

1.—*Show that when a quarter-wave transmission line of surge impedance Z_o is connected to a resistance R_E ohms at the receiver's end, the input impedance is $\frac{Z_o^2}{R_E}$.*

Answer.—The fundamental general equations for a no-loss transmission line of length l km. are:

$$\left.\begin{aligned}\mathfrak{V}_1 &= \mathfrak{V}_2 \cos \alpha l + jZ_o \mathfrak{J}_2 \sin \alpha l\\ \mathfrak{J}_1 &= j\,\mathfrak{V}_2 \frac{1}{Z_o} \sin \alpha l + \mathfrak{J}_2 \cos \alpha l\end{aligned}\right\}$$

where $\alpha = \omega\sqrt{LC} = \frac{\omega}{c} = \frac{\omega}{3 \times 10^5}$ radian per km.

and $\lambda = \frac{2\pi}{\alpha} = \frac{c}{f}$

Since, in this case, $\mathfrak{V}_2 = \mathfrak{J}_2 R_E$, the input impedance is

$$\mathfrak{Z}_1 = \frac{\mathfrak{V}_1}{\mathfrak{J}_1} = \frac{\mathfrak{J}_2 R_E \cos \alpha l + jZ_o \mathfrak{J}_2 \sin \alpha l}{j\mathfrak{J}_2 \frac{R_E}{Z_o} \sin \alpha l + \mathfrak{J}_2 \cos \alpha l}$$

and since $l = \frac{\lambda}{4}$ so that $\alpha l = \frac{\pi}{2}$ it follows that

$$\mathfrak{Z}_1 = \frac{Z_o^2}{R_E}$$

2.—*What methods are used for connecting a radio sending station to the antenna? What are the characteristic features of each method?*

Answer.—For modern transmitting stations the antenna is sufficiently far removed from the station to ensure that no disturbing reaction from the radiated field can reach it. A further consideration is that in the event of the mast collapsing the station building shall not be damaged by the scattering of the fractured parts. In the case of short-wave and ultra-short wave transmission the distance between station and antenna is usually necessarily large, since the dipoles for directional antenna are generally fixed at a great height above the earth.

For these reasons the coupling of antenna and sending station must be effected by means of a non-radiating energy feeder system. Three types of such coupling lines are used, viz :

(i) A single feeder line with earth return, (ii) a system of two parallel lines, (iii) a concentric cable.

The method (i) requires very exact matching, as otherwise standing waves will develop so that the feeder will then radiate energy which may interfere with the radiated energy from the antenna. In view of this disadvantage this method is seldom used. The method (ii) is widely used, and is not likely to give serious difficulty due to radiation. In this method the two wires are arranged to lie parallel to one another at the same height above the earth and as close together as is reasonably safe from the point of view of insulation security, also the supporting masts are sufficiently close together to avoid the possibility of swinging of the suspended wires and so altering the capacitance to earth. The two lines are held at the same distance apart by suitably spaced distance ties. Since one line is the return path for the other, the current in the two wires are flowing in mutually opposite directions, and so largely neutralise each other's radiation.

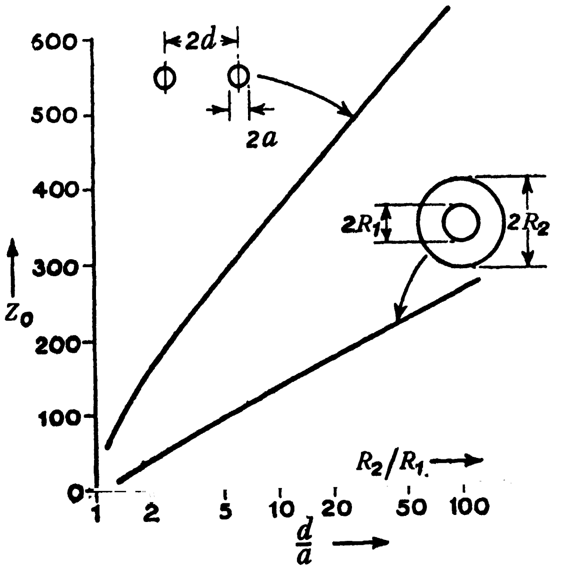


Fig. 1

Fig. 1 gives the surge impedance for a transmission system of two parallel wires (Figs. 2 and 3) and a co-axial cable (Fig. 4), as a function of the ratio $\left(\frac{\text{distance between wires}}{\text{radius of each wire}}\right)$ as calculated from the expression :

$$Z_o = 277 \log_{10} \left\{ \frac{d}{a} \left[1 + \sqrt{\left(1 - \frac{a}{d}\right)2} \right] \right\} \text{ ohms} \ldots\ldots\ldots \text{(i)}$$

The method of calculating the damping factor β is as follows:
(i). Taking the numerical data from the overhead line of Fig. 2,

$$2d = 11.4 \text{ cm.}: \quad 2a = 0.6 \text{ cm.}: \quad \frac{d}{a} = 19$$

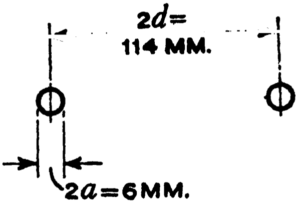


Fig. 2

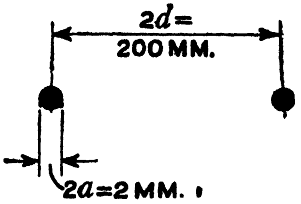


Fig. 3

and applying the formula (i) for calculating the surge impedance, it is found that

$$Z_o = 457\,\Omega$$

For a frequency, $f = 10^6$ hz.,

$$Y = 2\sqrt{2}\,\frac{\pi\sqrt{f}}{\sqrt{\rho \times 10^9}} \text{ (see page 442, “ Principles ”)}$$

where, $\rho = 1.77 \times 10^{-6}$ ohm per cm. cube for copper wire.

Then, $$Y = \frac{2\sqrt{2}\cdot\pi \times 10^3}{\sqrt{1{\cdot}77 \times 10^3}} = \frac{8{\cdot}9 \times 10^3}{42} = 210$$

and $$Y.a = 210 \times 0{\cdot}3 = 63$$

Since $Y.a > 50$, it is permissible to use the following expression (see page 454 of “ Principles ” Chapter XIV) for the resistance to a.c., viz.,

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = \frac{a}{2\sigma}$$

where

$$\sigma = 5{,}030\sqrt{\frac{\rho}{f}}$$ and is the effective depth of penetration, that is,

$$\sigma = 0.0067 \text{ cm.}$$

The resistance of the conductor to d.c. is

$$R_{d.c} = \frac{\rho \times 10^5}{\pi . a^2} = \frac{0.177}{\pi \times 0.3^2} = 0.622 \text{ ohm per km. of wire.}$$

The resistance per km. run of the transmission line, that is, for the " go" and " return " conductors in series is

$$R_{d.c.} = 1.24 \text{ ohm per km. run}$$

Hence,

$$R_{a.c.} = R_{d.c.}\left(\frac{a}{2\sigma}\right) = 1.24\left(\frac{0.3}{2 \times 0{\cdot}0067}\right) = 27.8\ \Omega/\text{km.}$$

and the damping factor is

$$\beta = \frac{R_{a.c}}{2} \cdot \frac{1}{Z_o} = \frac{13{\cdot}9}{457}.$$

that is

$$\beta = 0{\cdot}03 \ \underline{\text{neper per km.}} \text{ run of the line.}$$

(ii). Taking the numerical data of the overhead line of Fig. 3,

$$d = 10 \text{ cm.} : a = 0{\cdot}1 \text{ cm.} : \frac{d}{a} = 100$$

and applying the formula (i), it is found that,

$$Z_o = 570 \text{ ohms.}$$

For a frequency $f = 10^6$ hz.,

$Y = 210$ as in (1)

and $Y.a = 21$

Since $Y.a > 2.82$, the following expression may be used to find the value of the a.c. resistance (page 454, " Principles " Chapter XIV), viz.,

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = 0.325\ Y.a + 0.25$$

so that $R_{a.c.} = R_{d.c} \times 7.1.$

The resistance to d.c. is

$$R_{d.c.} = \frac{\rho \times 10^5}{\pi . a^2} \text{ ohms per km. of wire}$$

that is

$$R_{d.c} = 5.62 \text{ ohms per km. of wire.}$$

The resistance to d.c. per km. run of the transmission line, that is, for the " go " and " return " conductors, is

$$R_{d.c} = 2 \times 5.62 = 11.24 \text{ ohms per km. run of the line.}$$

The resistance of the line to a.c. is therefore

$$R_{a.c} = 11{\cdot}24 \times 7{\cdot}1 = 80\Omega \text{ per km. run.}$$

The damping factor is therefore

$$\beta = \frac{R_{ac}}{2}\,\frac{1}{Z_0} = \frac{40}{570} = \underline{0\cdot07 \text{ neper/km.}}$$

A feeder system of two such parallel lines is known as a " Lecher Line " (see, also, Example 4, Test Paper, Chapter XV).

The method (iii) which makes use of coaxial cables is completely free from radiation. The damping quality is seen from the numerical data given in Fig. 4.

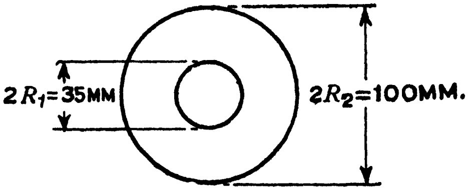


Fig. 4

The resistance to alternating current can be calculated from the formula already given on page 456, Chapter XIV, of " Principles," viz.,

Resistance to a.c. for a copper core and a copper sheath is,

$$R_{a\,c} = 41\cdot2 \times 10^{-4}\left\{\frac{1}{R_1} + \frac{1}{R_2}\right\}\sqrt{f} \text{ ohms/km.}$$

Inserting $R_1 = 1\cdot75$ cm. : $R_2 = 5\cdot0$ cm. : $f = 10^6$ hz.

$$R_{a\,c.} = 41\cdot2 \times 10^{-4}\left\{\frac{1}{1\cdot75} + \frac{1}{5\cdot0}\right\} 10^3 \text{ ohms per km.}$$

$$= 41.2 \times 10^{-1}\,\{0.57 + 0.2\} \text{ ohms per km.}$$

that is $\qquad R_{a\,c.} = 3\cdot17$ ohms per km.

The expression for the surge impedance is (see page 456)

$$Z_0 = 138 \log_{10}\frac{R_2}{R_1} \text{ ohms}$$

$$= 138 \log_{10}\frac{5}{1\cdot75} \text{ ohms}$$

that is $\qquad Z_0 = 138 \times 0\cdot5 = \underline{69 \text{ ohms.}}$

The damping factor is then (see Test Paper, Chapter XIV, Example 18, expression (iv), page 282,

$$\beta = \frac{R_{a.c.}}{2}\cdot\frac{1}{Z_0} = \frac{1.58}{69}$$

that is $\qquad \beta = 0.023$ neper per km.

The value of the surge impedance as a function of the ratio $\frac{R_2}{R_1}$ is shown in Fig. 1.

Since radiation from the feeder system must be avoided at all costs, so that the whole of the energy which is supplied to the feeder shall be delivered to the antenna, it is necessary that the feeder shall be matched to the sending end of the line and also to the antenna. Such matching can be obtained by tuning the feeder or by means of an air-core coupling. transformer at each end of the feeder (see also, Example 7 of this Test Paper), or by a combination of both types of matching.

3.—If a quarter-wave transmission line is open-circuited at the receiver's end, find what the input impedance will be if the supply frequency is slightly higher than the resonance value.

Answer.—From Example 1 of this Test Paper the input impedance of a no-loss transmission line l km. long is seen to be:

$$\mathfrak{Z}_1 = \frac{\mathfrak{B}_1}{\mathfrak{I}_1} = \frac{1 + j\frac{Z_0}{\mathfrak{Z}_2}\tan \alpha l}{\frac{1}{\mathfrak{Z}_2} + j\frac{1}{Z_0}\tan \alpha l}$$

When the line is open-circuited at the receiver's end $\mathfrak{Z}_2 = \infty$ so that,

$$\mathfrak{Z}_1 = -j\,Z_0 \cot \alpha l$$

and for a line which is exactly one-quarter wavelength,

$$\alpha l = \alpha\frac{\lambda}{4} = \frac{\pi}{2} \text{ so that } \mathfrak{Z}_1 = 0 : \text{ (see Fig. 5).}$$

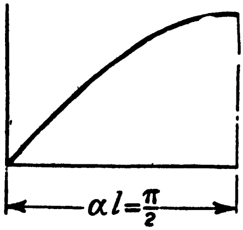


Fig. 5

If the frequency is slightly higher than the resonance frequency ω_0 this means that the line is correspondingly longer than $\frac{\lambda}{4}$, in consequence of which the input impedance will attain a definite magnitude. Thus, if the length of the line is $(l + \triangle l)$ where $l = \frac{\lambda}{4}$ and, noting that

$$\alpha = \omega_0 \sqrt{LC} = \frac{\omega_0}{c},$$

then
$$\alpha\, l = \alpha \frac{\lambda}{4} = \frac{\omega_0}{c}\, l = \frac{\pi}{2}$$

and
$$\triangle\, \alpha\, l = \frac{\triangle\omega_0}{\omega_0}\frac{\pi}{2}$$

The input impedance is therefore

$$(\mathfrak{Z}_i)_{o\cdot c} = -j\, Z_0 \cot\,(\alpha\;\; l + \triangle\,\alpha\, l) = j\, Z_0 \tan \frac{\triangle\omega_0}{\omega_0}\frac{\pi}{2}$$

that is,
$$(\mathfrak{Z}_i)_{o\cdot c} \doteqdot j\, Z_0 \frac{\triangle\omega_0}{\omega_0}\frac{\pi}{2} : \text{ when } \frac{\triangle\omega_0}{\omega_0} \text{ is small.}$$

If the line has a resistance of R ohms per km., it can be shown that for a frequency which exceeds the resonance value by the small amount $\triangle\omega_0$

$$(\mathfrak{Z}_i)_{o\cdot c} = \frac{R}{2}\, l \left\{ 1 + j \frac{\sqrt{\frac{L}{C}}}{\frac{R}{2}\, l} \frac{\pi}{2} \frac{\triangle\omega_0}{\omega_0} \right\} \text{ohms, where } l = \frac{\lambda}{4}.$$

For absolute resonance : $\mathfrak{Z}_i = \frac{R}{2}\, l = \frac{R}{2}\frac{\lambda}{4}$ ohms.

As a numerical example of these results, a co-axial cable will be considered of which the dimensions are as follows (see Fig. 6) :

$$\mathrm{D}_2 = 7{\cdot}5 \text{ mm.} : \mathrm{D}_1 = 2{\cdot}1 \text{ mm.} : \frac{D_2}{D_1} = 3{\cdot}6$$

$$L = 2 \left[\log_e \frac{D_2}{D_1}\right] 10^{-9} \text{ henry per cm.}$$

$$C = \frac{1}{1{\cdot}8} \frac{1}{\log_e \frac{D_2}{D_1}} 10^{-12} \text{ farad per cm.}$$

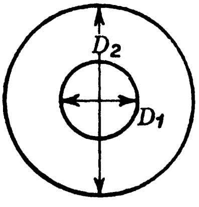


Fig. 6

$$Z_0 = \sqrt{\frac{L}{C}} = 60 \log_e 3{\cdot}6 = 76{\cdot}5 \text{ ohms.}$$

Conductivity $\varkappa = 57 \times 10^4$ siemens/cm./cm².

$$R = \left(\frac{2}{D_1} + \frac{2}{D_2}\right)\sqrt{\frac{f}{\sigma}10^{-9}} \text{ ohm per cm.}$$

and, for $f = 6 \times 10^9$ hz.

$$R = \left(\frac{2}{0{\cdot}21} + \frac{2}{0{\cdot}75}\right)\sqrt{\frac{6}{57 \times 10^4}} = 0{\cdot}04 \text{ ohm per cm.}$$

The damping factor is

$$\beta = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{0{\cdot}02}{76{\cdot}5} \text{ neper per cm.} = 26 \text{ neper per km.}$$

The input impedance when the cable is open-circuited at the receiver's end and $l = \frac{\lambda}{4}$ is $(\mathfrak{Z}_1)_{oc} = \frac{R}{2}l = \frac{R}{2}\cdot\frac{\lambda}{4} = 0{\cdot}02 \times 1{\cdot}25 = 0{\cdot}025$ ohm.

since for $f = 6 \times 10^9$: $\lambda = \frac{c}{f} = \left(\frac{3 \times 10^5}{6 \times 10^9}\right)$ km. $= 5$ cm.

If the frequency is 2 per cent. greater than the resonance value so that

$$\frac{\Delta\omega}{\omega_0} = 0{\cdot}02$$

then
$$j\frac{\sqrt{\frac{L}{C}}}{\frac{R}{2}l}\,\frac{\pi}{2}\,\frac{\Delta\omega}{\omega_0} = j\,96$$

If $\frac{\Delta\omega}{\omega_0} = 0{\cdot}001$, i.e., 0·1 per cent., so that $\Delta\omega = 6 \times 10^6$ hz.

then
$$j\frac{\sqrt{\frac{L}{C}}}{\frac{R}{2}l}\,\frac{\pi}{2}\,\frac{\Delta\omega}{\omega_0} = j\,4{\cdot}8$$

so that the input impedance is then $\sqrt{1 + 4{\cdot}8^2} = 4.9$ times the value at absolute resonance.

4.—If a quarter-wave transmission line is short-circuited at the receiver's end, find what will be the impedance when the frequency is slightly higher than the resonance value.

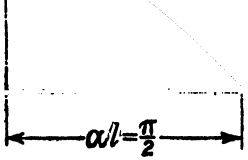


Fig. 7

Answer.—The expression given in Example 1 of this Test Paper for the input impedance of a non-loss transmission line may be re-written in a more convenient form for the purposes of the present example, viz. (see also Fig. 7).

$$\mathfrak{Z}_1 = \frac{\mathfrak{Z}_2 + j Z_0 \tan \alpha l}{1 + j \dfrac{\mathfrak{Z}_2}{Z_0} \tan \alpha l}$$

so that, for $\mathfrak{Z}_2 = 0$,

$$(\mathfrak{Z}_1)_{s\,c} = \underline{j Z_0 \tan \alpha l)}$$

For the resonance frequency, $\alpha l = \frac{\pi}{2}$: $\underline{(\mathfrak{Z}_1)_{s.c.} = \infty}$

For a frequency $\omega_0 + \triangle\omega : \alpha l = \frac{\pi}{2} + \frac{\triangle\omega}{\omega_0} \frac{\pi}{2}$

so that

$$(\mathfrak{Z}_1)_{s.c.} = -j Z_0 \cot \frac{\triangle\omega}{\omega_0} \frac{\pi}{2} = -j Z_0 \frac{1}{\tan \dfrac{\triangle\omega}{\omega_0} \dfrac{\pi}{2}}$$

which, for small values of $\frac{\triangle\omega}{\omega_0}$ gives

$$(\mathfrak{Z}_1)_{s.c.} = -j Z_0 \frac{\omega_0}{\triangle\omega} \frac{2}{\pi}$$

When the resistance of the line is R ohm per km., it can be shown that the input impedance is

$$(\mathfrak{Z}_1)_{s\,c} = \frac{\frac{L}{C}}{\frac{R}{2}l} \left\{ 1 - j \frac{\sqrt{\frac{L}{C}}}{\frac{R}{2}l} \frac{\pi}{2} \frac{\triangle\omega}{\omega_0} \right\}$$

for a frequency which exceeds the resonance value ω_0 by the small amount $\triangle\omega$.

At absolute resonance the input impedance is

$$(\mathfrak{Z}_1)_{s.c.} = \frac{L}{C} \frac{2}{R \cdot l} \text{ ohms.}$$

Applying this result to the numerical data for the co-axial cable of Example 3 of this Test Paper, then at absolute resonance,

$$(\mathfrak{Z}_1)_{s.c.} = \frac{L}{C} \frac{2}{R \cdot l} = Z_0^2 \frac{8}{R\lambda} = \frac{76{\cdot}5^2 \times 8}{0{\cdot}04 \times 5} = 0{\cdot}235 \times 10^6 \text{ ohms}$$

For the case in which the frequency is 0·1 per cent. greater than resonance, the j term in the expression $(\mathfrak{Z}_1)_{\text{s c.}}$ is

$$j\frac{\sqrt{\frac{L}{C}}}{\frac{R}{2}l}\,\frac{\pi}{2}\,0{\cdot}001 = j\,\frac{76{\cdot}5 \times 1{\cdot}57}{0{\cdot}02 \times 1{\cdot}25 \times 10^3} = j\,4{\cdot}8$$

so that the input impedance is then $\{\sqrt{1+4{\cdot}8^2|}\} = 4{\cdot}9$ times the value at absolute resonance, that is $= 1{\cdot}15 \times 10^6$ ohms:

It is further to be noted with regard to this co-axial cable that,

$$L = 2\left\{\log_\epsilon \frac{R_2}{R_1}\right\} 10^{-9} = 2{\cdot}56 \times 10^{-9} \text{ henry per cm.}$$

so that $\omega_0 \cdot L = (2\pi \times 6 \times 10^9)(2{\cdot}56 \times 10^{-9}) = 97$ ohms

and $R = 0{\cdot}04$ ohm. (See also Example 3 of this Test Paper.)

hence

$$\frac{R}{\omega_0 L} = 0{\cdot}00041 = \frac{1}{Q}$$

Also, since : f $= 6 \times 10^9$ hz : $\alpha = \frac{\omega_0}{c} = \omega_0\sqrt{LC} : \beta = \frac{R}{2}\sqrt{\frac{C}{L}}$

it follows that,

$$\frac{\beta}{\alpha} = \frac{R}{2\omega L} = 0{\cdot}000205$$

and

$$Q = \frac{\omega L}{R} = 2{,}440$$

5.—*Compare the characteristics of the normal high-frequency cable and the co-axial cable.*

Answer.—Overhead transmission lines for light current purposes are being superseded continually by underground cables in order to eliminate the interference from outside electrical disturbances. Actual measurements on such cables at high frequencies have shown that, whilst the surge impedance vector is practically an entirely "real" quantity and almost constant for a wide range of frequencies, the damping factor β rapidly increases with the frequency. For example, for a multi-strand cable the damping factor for strands of 1.4 mm. diameter at 100,000 hertz is $\beta = 0{\cdot}2$ neper per km., and at 1,000,000 hertz $\beta = 1$ neper per km.

In Fig. 8 is shown a normal type of multi-strand cable for long-distance high frequency transmission. It will be seen that the distance apart of two wires of a pair is necessarily very small, so that the capacitance is relatively high and the inductance relatively small. In accordance, however, with the expression

$$\beta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

a small damping requires small capacitance and large inductance (See also Example 12, page 170), so that in order to reduce the damping it is necessary to separate the two strands of a pair. Owing primarily to the requirements of television, a new type of cable has been evolved on these lines, which will permit very high frequency operation with a relatively very small damping factor. One method for doing this is shown in Fig. 9, in which the two strands *a* and *b* are arranged within a screened sheath and held in position by means of spacing supports of insulating material.

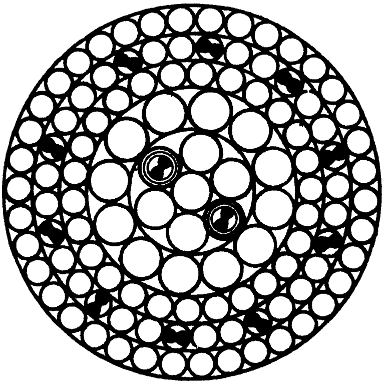


Fig. 8

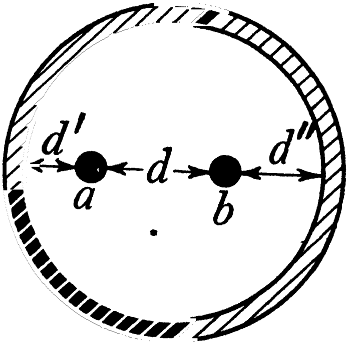


Fig. 9

It is to be observed that the distance *d* between the two strands must be as great as possible and at the same time the strands must not be too close to the sheath. The energy loss in such special high frequency cables is kept low by the use of a new type of synthetic insulating material, e.g. Styroflex (a development of "trolitul," see also pages 68 and 114, "Principles"). The damping factor β and the phase displacement angle α vary with the frequency, and when very high frequency bands are being transmitted the difference of the values of β and α respectively at the highest and lowest frequencies may reach seriously large magnitudes. In such cases it is necessary to connect the lines through a four-pole unit in order to neutralise the damping. The effect of the change of α in the line is to alter the relative time displacement of the several frequencies and when desirable this can be corrected by means of a cross-connected four-pole unit.

6.—Prove that the surge impedance of open space is 376.7 ohms.

ANSWER.—Fig. 10 shows a portion of the plane front of an electro-magnetic wave. It follows, then, from equation (7), page 496 "Principles," that

$$\oint (\mathfrak{E} \text{ el. m. units per cm.})\, dl = \mu . l \frac{dx}{dt} (\mathfrak{H} \text{ el. m. units}).$$

For the integration round the rectangle $NGG'N'$,

$$(l_{\text{cm.}}) (\mathfrak{E} \text{ el. m. units per cm.}) = (l_{\text{cm.}}) \frac{dx}{dt} (\mathfrak{H} \text{ el. m. units}) \qquad \text{(i)}$$

since μ in el. m. units $= 1$ for open space.

Similarly from equation (9), page 497, "Principles"

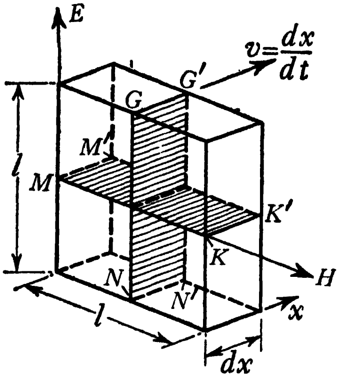


Fig. 10

$$(\mathfrak{H} \text{ el. m. units}) = \frac{1}{c^2} (\mathfrak{E} \text{ el. m. units per cm.}) \frac{dx}{dt} \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

and, by multiplying (i) and (ii) together, the speed of propagation is obtained, viz.,

$$v = \frac{dx}{dt} = 2{\cdot}9979 \times 10^{10} \text{ cm. per sec.} = c$$

Again, from page 218 of "Principles of Electrical Engineering,"

$$H \text{ el. m. units} = H \text{ oersted} = 4\pi \frac{I}{10} \text{ ampere per cm.},$$

and, noting that E el.m. units per cm. $= (V$ volts per cm.$)\, 10^8$ and substituting in (i), it is found that,

$$V \text{ (volts per cm) } 10^8 = (4\pi \frac{I}{10} \text{ amperes per cm.}) \times c$$

or
$$Z_0 = \frac{V \text{ volts/cm.}}{I \text{ amperes/cm.}} = \frac{4\pi\, c}{10^9} = 4\pi \times 29{\cdot}979 \text{ ohms.}$$

That is, the surge impedance of open space is $Z_0 = 376{\cdot}728$ ohms.

7.—Show how a transmission line may be matched to a load by means of a transformer.

Answer.—Page 335 of "Principles" shows how to derive the circuit which shall be electrically equivalent to the actual coupled circuit system of a transformer. In Fig. 11 two coupled circuits 1 and 2 are shown, and it is assumed that the resistance of each circuit is negligibly small, and also that the coupling is perfect so that,

$$M^2 = L_1 L_2,$$

that is to say, there is no electromagnetic leakage. If
w_1 is the number of turns in winding 1,
w_2 is the number of turns in winding 2,

$$u = \frac{w_1}{w_2} \text{ is the transformation ratio,}$$

then the equivalent simple series circuit of Fig. 12 is electrically identical with the coupled circuit of Fig. 11. If then a transmission line of surge impedance Z_{o1} ohms is connected through a transformer to a resistance load R_2 ohms, and if

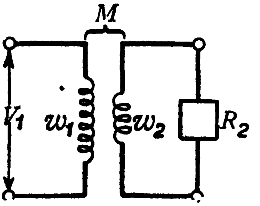


Fig. 11

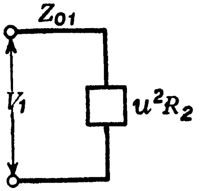


Fig. 12

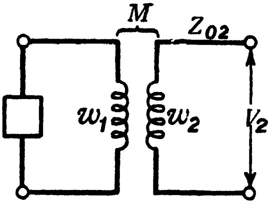


Fig. 13

$$Z_{o1} = u^2 R_2 \text{ ohms,}$$

the load will be matched to the line, and there will be no reflection effects produced at the junction of line and transformer.

Similarly, if the line of surge impedance Z_{o2}, Fig. 13, is connected to the resistance load of R_1 ohms through the transformer of ratio $\frac{1}{u}$ where $u = \frac{w_1}{w_2}$, then this resistance load will be matched with the line if,

$$Z_{o2} = \frac{1}{u^2} R_1 \text{ ohms.}$$

8.—What considerations of the damping characteristics of telephone lines govern the suitable spacing of the amplifiers ?

Answer.—The waves of current and pressure become attenuated as they travel along the line, and in order to restore their amplitude, amplifiers are connected at suitably spaced intervals. For long-distance telephony, the spacing may be 75 or 150 km., according to the damping. When very high frequency is used, the line damping becomes very large so that the distance between successive amplifiers must be reduced correspondingly. The ratio of the volt-amperes, pressure, or current at any point in the line to the corresponding value at some datum point, e.g., the sending end of the line, is a measure of the damping. In view of the possibility of interference effects such as "cross-talk" between two lines, the damping for long-distance telephone lines should be not greater than 3 nepers. In Fig. 14 is shown an example of such damping and amplification relationships. For transmission at very high frequencies the construction of the cables is such that any risk of cross-talk is practically eliminated so that the damping may then be allowed to reach the value of about 7 nepers before amplification becomes necessary.

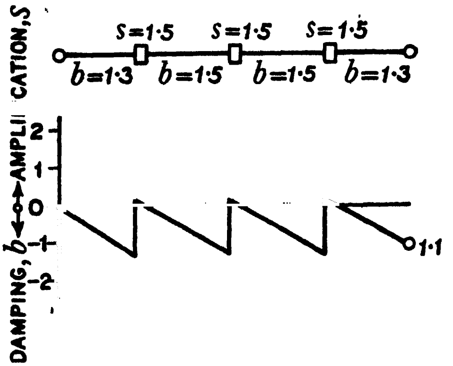


Fig. 14

The amplification at each successive section end of the line is only sufficient to make good the loss by damping in the previous section (see Fig. 14). If a greater amplification were to be introduced, the pressure would be raised to such an extent that it might give rise to interference with neighbouring lines. The natural logarithm of the ratio of the pressure, or of the current at the receiver's end of the line to the corresponding value at the sending end of the line is a measure of the residual damping. When a large number of lines have to be dealt with, the amplifiers are all assembled in one station and the distance between consecutive stations is about 75 km. For high frequency cables however, the amplifiers must be very much closer together.

9.—In the development of the Maxwell-Hertz equations (page 497 of "Principles of Electrical Engineering") it was assumed that the current density was uniform throughout the dipole. For practical attennae, however, this assumption is not valid. Find the height of an antenna, having uniformly distributed current density, which will be equivalent to the actual antenna having sinusoidally distributed current density.

ANSWER.—Let H be the geometrical height of the antenna and let the current density at any distance s from the upper end of the antenna be i_s so that (see Fig. 15.)

$$i_s = i_0 \sin \alpha s$$

Where i_0 is the current density at the ground end. Then for a quarter-wave antenna

$$\alpha . s = \frac{\pi}{2} \text{ for } H = s\text{: so that } = \alpha \frac{\pi}{2H}$$

The mean current density is :

$$i_{av} = \frac{1}{H}\int_0^H i_s \, d s = \frac{1}{H}\left[\frac{i_0}{\alpha}\cos \alpha s\right]_0^{H}$$

that is $$i_{av} = \frac{i_0}{H\alpha} = \frac{i_0 2H}{H\pi} = \frac{2}{\pi} i_0$$

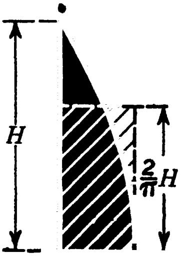


Fig. 15

If h is the height of the equivalent antenna having uniform current density $i_{av} = i_0$, then the current in the actual and equivalent antenna will be the same if $$i_0 h = i_0 \frac{2}{\pi} H$$

that is, if $$h = \frac{2}{\pi} H$$

10.—*Compare the characteristic features of high-frequency cables and of high-frequency radio for alternating current transmission.*

Answer.—High-frequency cables are used for a great variety of technical purposes, e.g., television, multiple-channel telephony, antennæ leads at both sending and receiving stations, delaying cables for impulse pressure equipments, feeder cables for high-frequency furnaces (in many such furnaces the frequency required is in the audio range).

It is a remarkable fact that even 10–12 years ago, the transmission of high-frequency alternating current by cables was not contemplated by high-frequency technicians, the primary stumbling block being the question of damping. When it is observed (see also Examples 16, 17 and 19, Test Paper, Chapter XV) that a paper-insulated long-distance double line with conductors of copper 1 mm. diameter at 1 Mhz frequency gives a damping of more than 1 neper per km., so that, for example, if $\beta = 1.15$ neper per km., the power at a distance of 3 km. from the sending end will be given by the expression

$$Ws = W_1 e^{-2\beta s} = W_1 e^{-2\times 3\cdot 45} \simeq W_1 e^{-6\cdot 9} = 0\cdot 001 W_1$$

so that the power has become reduced to one-thousandth part of its original value, and at a distance of 6 km. from the sending end the power will be

$$Ws = W_1 e^{2\times 6\cdot 0} = W_1 \times 10^{6}$$

so that in this case the power will have become reduced to one-millionth part of its original value. It is thus easily understood that transmission of high-frequency current by means of cables would appear to be out of the question. In consequence of such considerations, cables were only used until comparatively recently, for the transmission of energy at frequencies not greater than 60 hz, and the transmission of signals and music up to 10,000 hz. If it had not been for the fact that quite unexpectedly the development of television called for transmission circuits operating at frequencies of 10^6 hz and higher, cables might quite well have remained tied to their restricted low-frequency purposes. The question of multi-channel telephony had indeed already awakened interest in transmission at high frequencies and the "loaded" (see Chapter VII, page 192, "Principles of Electrical Engineering") submarine cable which connects Kay West to Havana, which was laid in 1930–31, may be regarded as the forerunner of this field of application. However, but for the demands of television the required urge would have been wanted for the development of very high frequency cables, and this type has become a practical commonplace during the last decade.

The primary characteristics of this development are the greatly reduced value of the damping factor which has been obtained (about 0.2 neper/km.); the reduction of the spacing distance of the amplifiers, and the stabilisation and increase in the degree of amplification of the amplifiers. In general, very high frequency cables are constructed with a concentric conductor (see also Example 2 of this Test Paper), in which special care is taken with respect to the dielectric characteristic of the insulating material (e.g., "trolitul," see Chapter IV, Table II, page 114).

The question arises as to what significance for future developments must be attached to the long-distance high-frequency cable. In television, just as in radio, two kinds of transmission possibilities are available—the wireless and the wired. As is well known, radio transmission in its early stage of development was almost exclusively propagated through open space, and it was only during the second decade of radio development that wired transmission was employed because of its greater freedom from disturbances. For television, the requirements are quite special. The range of ultra-short wave transmission is given by Prof. Tank, of Zurich, as about

$$100\sqrt{h} \text{ km.}$$

where h km. is the height of the antenna above the surface of the ground. If the antenna is sufficiently high, transmission can be operated over great distances, but disturbances due to the "shadow effect" caused by the unevenness of the ground in hilly districts may be serious. In any case, a television receiver can only receive *one* wireless transmitted programme, and the stringent limitations regarding the use of a permissible wave-band must be kept in mind. For such reasons, serious consideration had to be given to cable transmission for television, and this implies the provision of a high-frequency cable network in large cities. It is of interest to note that the R.C.A. has installed equipment to link New York and Philadelphia in which directed decimetre waves are transmitted over a distance of about 50 miles with an intermediate amplifying equipment, and this has given satisfactory results.

The future significance of high-frequency cables cannot yet be assessed with any certainty. Laboratory research will provide the basis on which the technical decision will have to be made as to whether, for any given ultra-high-frequency purpose, cables should be used.

The transmission of a directed beam of ultra-high-frequency waves in the range from 10 cm. to 1 cm. has the following advantageous features :

(1) Great distances can be bridged with relatively small power and consequently with small and light accessory apparatus.

(2) The antenna structure is simple and the equipment for mobile stations is light and compact.

(3) The waves are practically free from " fading," seasonal effects, and day or night effects.

(4) The amount of material for the station structure is relatively small.

It is to be observed that, although ultra-high-frequency technique has an astonishing record of achievement, it is still in process of development, and all the available evidence shows that still greater progress may be expected in the immediate future.

11.—Prove that if a parallel resonance circuit of inductance L henry and capacitance C farad and, if a non-inductive series resistance of R ohms is placed in each of the two parallel branches, then the input impedance of the system will be independent of the frequency if $R = \sqrt{\frac{L}{C}}$.

(Television Aerial of the Empire State Building, New York.)

ANSWER.—The input impedance $\mathfrak{Z}i$, Fig. 16, is given by

$$\frac{1}{\mathfrak{Z}_i} = \frac{1}{R + j\omega L} + \frac{1}{R - j\frac{1}{\omega C}} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad \text{(i)}$$

Differentiating this equation with respect to ω and equating to zero gives

$$\frac{d}{d\omega}\left(\frac{1}{\mathfrak{Z}_i}\right) = 0 = -\frac{jL}{(R + j\omega L)^2} - \frac{j\frac{1}{\omega^2 C}}{(R - \frac{1}{\omega C})^2}$$

$$L\left[R^2 - \mathrm{j}\frac{2R}{\omega C} - \frac{1}{(\omega C)^2}\right] = -\frac{1}{\omega^2 C}\left[R^2 + \mathrm{j}2\omega LR - (\omega L)^2\right] \text{........(ii)}$$

Equating respectively the real and the imaginary parts of this equation then gives

$$\left.\begin{aligned} LR^2 - \frac{L}{(\omega C)^2} &= -\frac{R^2}{\omega^2 C} + \frac{L^2}{C} \\ -\mathrm{j}\frac{2L\cdot R}{\omega\cdot C} &= -\mathrm{j}\frac{2L\cdot R}{\omega\cdot C} \end{aligned}\right\} \text{........................ (iii)}$$

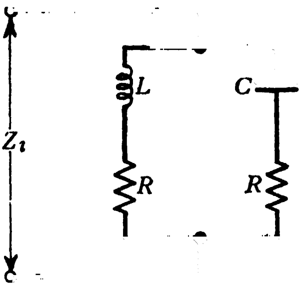


Fig. 16

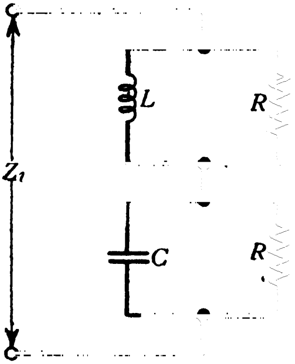


Fig. 17

that is to say, the imaginary parts disappear from equation (i) and the equation for the real parts may be written

$$R^2\left(L + \frac{1}{\omega^2 C}\right) = \frac{L}{(\omega\cdot C)^2} + \frac{L^2}{C} \text{ .... (iv)}$$

If then the frequency is such as to bring L and C into resonance, that is, if

$$\omega L = \frac{1}{\omega C}$$

then,

$$R^2\left(\frac{2}{\omega^2 C}\right) = 2(\omega L)^2 L$$

so that,

$$R^2 = \frac{L}{C}(\omega L)^2\omega^2 C^2$$

that is $$R = \sqrt{\frac{L}{C}} \tag{v}$$

Hence, if L and C are in resonance and if $R = \sqrt{\frac{L}{C}}$ the impedance of the circuit of Fig. 16 is a purely real quantity and is therefore independent of the frequency. For these conditions, the value of the impedance is given by the equation (i), viz.,

$$\frac{1}{\mathfrak{Z}_i} = \frac{1}{\sqrt{\frac{L}{C}} + j\omega L} + \frac{1}{\sqrt{\frac{L}{C}} - j\frac{1}{\omega C}}$$

which reduces to,

$$\mathfrak{Z}_i = \sqrt{\frac{L}{C}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(vi)}$$

The same result holds good if a non-inductive resistance $R = \sqrt{\frac{L}{C}}$ is placed *in parallel* with L and C respectively as shown in Fig. 17. An important example of the application of this principle is found in the television aerial of the Empire State Building, New York, and this is shown diagrammatically in Fig. 18, the corresponding equivalent circuit being given in Fig. 19.

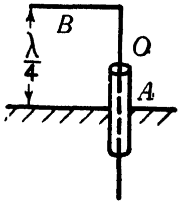


Fig. 18

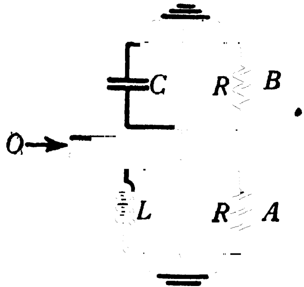


Fig. 19

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